ASYMPTOTIC LAWS FOR UPPER AND STRONG RECORD VALUES IN THE EXTREME DOMAIN OF ATTRACTION AND BEYOND

Abstract. Asymptotic laws of records values have usually been investigated as limits in type. In this paper, we use functional representations of the tail of cumulative distribution functions in the extreme value domain of attraction to directly establish asymptotic laws of records value, not necessarily as limits in type. Results beyond the extreme value value domain are provided. Explicit asymptotic laws concerning very usual laws are listed as well. Some of these laws are expected to be used in fitting distribution.
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1. Introduction

Let X, X_1, X_2, $\cdots$ be a sequence independent real-valued randoms, defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common cumulative distribution function F, which has the lower and upper endpoints, and the generalized inverse function respectively defined by

$$lep(F) = \inf\{x \in \mathbb{R},\ F(x) > 0\},\ uep(F) = \sup\{x \in \mathbb{R},\ F(x) < 1\}$$

and

$$F^{-1}(u) = \inf\{x \in \mathbb{R},\ F(x) \geq u\}\ for\ u \in]0,1[\ and\ F^{-1}(0) = F^{-1}(0+).$$

Finally, let us consider the sequence of strong record values $X^{(1)} = X_1$, $X^{(n)}$, $\cdots$ and the sequence of record times $U(1) = 1$, $U(2)$, $\cdots$

Before beginning an asymptotic theory, we should be sure that we have an infinite sequence $(X^{(n)})_{n\geq 1}$. For a bounded random variable with finite upper bound $uep(F)$ such that $\mathbb{P}(X = uep(F)) > 0$, we have $(X^{(n)} < uep(F))$

finitely often. This happens for classical integer-valued and bounded random variables as Binomial laws. In such cases, the asymptotic theory is meaningless. But, an interesting question would be the characterization the infinite random sequence $(n_k)_{k\geq 1}$ such that $X_{n_k} = uep(F)$ for all $k \geq 1$.

In all other cases, even if $uep(F)$ is bounded, the sequence $(X^{(n)})_{n\geq 1}$ is infinite. So, the results of this paper apply to *cdf*'s Fsuch that $\mathbb{P}(X = uep(F)) = 0$. In that context, asymptotic laws have been proposed in the literature by many authors like Tata (1969), Resnick (1987), Nevzorov (2001), etc., in relation with Extreme Value Theory, as limits in type in the form

$$(\exists (A_n)_{n\geq 1} \subset \mathbb{R}_+ \setminus \{0\}),\ \exists\ (B_n)_{n\geq 1} \subset \mathbb{R},\ \frac{X^{(n)} - B_n}{A_n} \rightsquigarrow Z, \tag{1.1}$$

where $\rightsquigarrow$ stands for the convergence in distribution and Z is a non-degenerate random variable. The motive beneath this search is the following. If we denote by $M(n) = \max(X_1, \cdots, X_n)$ as the n-th maximum for $n \geq 1$, it is clear that we have

$$\forall\ n \geq 1,\ X^{(n)} = M(U(n)). \tag{1.2}$$

Since for any F in the extremal domain of attraction $\mathcal{D}$, we have that for some $\gamma \in \mathcal{R}$,

$$(\exists (a_n)_{n\geq 1} \subset \mathbb{R}_+ \setminus \{0\}),\ (\exists (b_n)_{n\geq 1} \subset \mathbb{R},\ \frac{M(n) - a_n}{b_n} \rightsquigarrow Z_\gamma, \tag{1.3}$$

where the *cdf* of Z_γ is the Generalized Extreme Value distribution defined by

$$G_\gamma(x) = \exp(-(1+\gamma x)^{1/\gamma}),\ 1+\gamma x > 0,\ G_0(x) = \exp(-\exp(-x))\ for\ x \in \mathbb{R}.$$

In Extreme value Theory, Formula (1.3) is rephrased as *F is attracted by* G_γ denoted by $F \in D(G_\gamma)$.

From Formulas (1.2) and (1.3) and from the fact that $U(n) \to +\infty$ as $n \to +\infty$, the investigation of the validity of (1.1) was justified enough. The results of the cited authors and others were positive with the stunning result that the *cdf* of Z should be of the form $\Phi(g(x))$, $x \in \mathbb{R}$, where Φ is the *cdf* of the standard normal law and g satisfies one of three definitions (in which c is a positive constant)

$$g(x) = x,\ x \in \mathbb{R}.$$
$$g(x) = -\infty 1_{(x<0)} + (c \log x)\ 1_{(x\geq 0)}, x \in \mathbb{R}.$$
$$g(x) = (-c \log -x)\ 1_{(x<0)} + \infty\ 1_{(x>0)},\ x \in \mathbb{R}.$$

Instead of using this mathematically appealing approach based on functional equations, an other approach consisting in directly finding the asymptotic laws of $X^{(n)}$, not necessarily in the form of Formula (1.1) is possible and we proceed to it here. That approach is based on representations of $F \in \mathcal{D}$ of Karamata and de Haan for example.

Our achievement is the finding the asymptotic laws of the records for all $F \in \mathcal{D}$. First, for $\gamma \neq 0$, outside the frame Formula (1.1), that is as limits in type, and without any further condition. Secondly, for $\gamma = 0$, within the frame of Formula (1.1), under a general regularity condition. That regularity condition generally holds for usual *cdf*'s.

We also give general conditions to ensure the asymptotic normality of the records values for F not necessarily in the extremal domain. Finally, we give detailed asymptotic laws of the records of a list of remarkable *cdf*'s with specific coefficients.

In this paper we want short, we use many results from Extreme Value Theory and Records Values Theory. So, for more details, we refer the reader to the books of Ahnsanullah (1995), Nevzorov (2001), etc for an easy introduction to records and to those of Galambos (1985), de Haan (1970), Resnick (1987), Lo *et al.* (2018), etc. concerning Extreme Value Theory.

To finish this introduction, we recall two important tools of extreme value theory that form the basis of our method. The first is the following proposition. Suppose that $X \geq 0$, that is $F(0) = 0$. In that case, we define $Y = \log X$ with *cdf* $G(x) = F(e^x)$, $x \in \mathbb{R}$ and we have

Proposition 1. *(see Lo (1986)) We have the following equivalences.*

(1) If $\gamma > 0$,

$$F \in D(G_\gamma) \Leftrightarrow (G \in D(G_0) \text{ and } R(x,G) \to \gamma \text{ as } x \to uep(G)).$$

(2) If $\gamma = 0$,

$$F \in D(G_0) \Leftrightarrow (G \in D(G_0) \text{ and } R(x,G) \to 0 \text{ as } x \to uep(G)).$$

(3) If $\gamma < 0$,

$$F \in D(G_\gamma) \Leftrightarrow (G \in D(G_\gamma).$$

In the second place, we recall the following representations of *cdf*'s in the extreme value domain that repeatedly will be used in the sequel.

Proposition 2. *(karamata (1962) and de Haan (1970)) We have the following characterizations for the three extremal domains.*

(a) $F \in D(H_\gamma)$, $\gamma > 0$, *if and only if there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \to u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admits the following representation of Karamata*

$$F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{1.4}$$

(b) $F \in D(H_\gamma)$, $\gamma < 0$, *if and only if* $uep(F) < +\infty$ *and there exist a constant* c *and functions* $a(u)$ *and* $\ell(u)$ *of* $u \in]0,1]$ *satisfying*

$$(a(u), \ell(u)) \to (0,0) \text{ as } u \to 0,$$

such that F^{-1} *admit the following representation of Karamata*

$$uep(F) - F^{-1}(1-u) = c(1+a(u))u^{-\gamma} \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{1.5}$$

(c) $F \in D(H_0)$ *if and only if there exist a constant* d *and a slowly varying function* $s(u)$ *such that*

$$F^{-1}(1-u) = d + s(u) + \int_u^1 \frac{s(t)}{t} dt, 0 < u < 1, \tag{1.6}$$

and there exist a constant c and functions $a(u)$ and $\ell(u)$ of $u \in]0,1]$ satisfying

$$(a(u), \ell(u)) \rightarrow (0,0) \text{ as } u \rightarrow 0,$$

such that the function $s(u)$ of $u \in]0,1[$ admits the representation

$$s(u) = c(1+a(u)) \exp\left(\int_u^1 \frac{\ell(t)}{t} dt\right). \tag{1.7}$$

Moreover, if $F^{-1}(1-u)$ is differentiable for small values of s such that $r(u) = -u(F^{-1}(1-u))' = u\, dF^{-1}(1-u)/du$ is slowly varying at zero, then 1.6 may be replaced by

$$F^{-1}(1-u) = d + \int_u^{u_0} \frac{r(t)}{t} dt, 0 < u < u_0 < 1, \tag{1.8}$$

which will be called a reduced de Haan representation of F^{-1}.

The rest of the paper is organized as follows. The results are stated in Section 2. Examples and Applications are given in Section 3. The proofs are stated in Section 4. The computation related to examples in Section 3 are detailed in the Appendix Section 6. The paper closed by a conclusion in Section 5.

2. Results

Before we state our results, we recall that any $F \in \mathcal{D}$ is associated to a couple of functions $(a(u), b(u))$ of $u \in [0,1]$ as defined in the representations of Proposition 2 for $F \in D(G_\gamma)$, $\gamma \neq 0$. In the special case where $\gamma = 0$, the pair of functions $(a(circ), b(\circ))$ is used in the representation of the function $s(u)$ is f $u \in [0,1]$ in Representation (1.6).

We will need the following condition. Let us define for any $n \geq 1$ a finite sum of n standard exponential random variables

$$S_{(n)} = E_{1,n} + \cdots + E_{n,n},$$

denote

$$V_n = \exp(-S_{(n)}) \text{ and } v_n = \exp(-n),\ n \geq 1$$

and finally set the hypotheses

$$(Ha)\ :\ \sup\left\{\left|\frac{u}{v}-1\right|,\ \min(v_n,V_n)\leq u,v\leq \max(v_n,V_n)\right\}\rightarrow_{\mathbb{P}} 0\ as\ n\rightarrow+\infty,$$

$$(Hb)\ :\ (\exists\alpha>0),\ \sqrt{n}\ s(v_n)\rightarrow\alpha\ as\ n\rightarrow+\infty,$$

where $\rightarrow_{\mathbb{P}}$ stands for the convergence in probability.

Here are our results that cover the whole extreme value domain of attraction. For $\gamma\neq 0$, we need no condition.

Let us begin by asymptotic laws for $F\in\mathcal{D}$.

Theorem 1. *Let $F\in D(G_\gamma)$, $\gamma\in\mathbb{R}$. We have :*

(a) If $\gamma>0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{X^{(n)}}{F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}}\rightsquigarrow LN(0,\gamma^2),$$

where $LN(m,\sigma^2)$ is the lognormal law of parameters m and $\sigma>0$.

(b) If $\gamma>0$ and $X\geq 0$, $Y=\log X\in D(G_0))$ and $R(x,G)\rightarrow\gamma\ as\ x\rightarrow uep(G)$ and we have

$$\frac{Y^{(n)}-G^{-1}\left(1-e^{-n}\right)}{\sqrt{n}}\rightsquigarrow\mathcal{N}(0,\gamma^2).$$

(c) If $\gamma<0$, the asymptotic law of $X^{(n)}$ is lognormal, precisely

$$\left(\frac{uep(F)-X^{(n)}}{uep(F)-F^{-1}\left(1-e^{-n}\right)}\right)^{n^{-1/2}}\rightsquigarrow\exp(\mathcal{N}(0,\gamma^2)).$$

(d) Suppose that $\gamma=0$ and $R(x,G)\rightarrow 0\ as\ x\rightarrow uep(G)$. If (Ha) and (Hb) hold both, we have

$$X^{(n)}-F^{-1}\left(1-e^{-n}\right)\rightsquigarrow\mathcal{N}(0,\alpha^2).$$

More precisely, we have : Given $\gamma=0$, $R(x,G)\rightarrow 0\ as\ x\rightarrow uep(G)$ and (Ha), the above asymptotic normality is valid if and only if (Hb) holds.

Beyond distributions in $\mathcal{D}$, we may use the delta-method as follows. Drawing lessons from Theorem 1, we might be tempted to generalize point (a) by imposing that F^{-1} satisfies, for some coefficient γ,

$$\forall \lambda > 0,\ F^{-1}(1-\lambda u)/F^{-1}(1-u) = \lambda^{\gamma}(1+o(1)),\ u \in]0,1[.$$

But, by Extreme Value Theory, this would imply that $F \in G_{\gamma}$ and nothing new would happen. But trying a generalization from Point (c) would be successful. Let us define the following hypotheses :

(Ga) F is differentiable in some left neighborhood $]x_0,\ uep(F)[$ of $uep(F)$.

(Gb) The function

$$s(x) = e^{-x}\left[F^{-1}(1-t)\right]'_{t=e^{-x}},\ e^{x} < u_0 < 1,\ for\ some\ u_0 \in]0,1[$$

decreases to 0 as $x \to +\infty$ and is such that : for any sequence $(x_n, y_n)_{n\geq 1}$ such that

$$\limsup_{n\to+\infty} |x_n - y_n|/\sqrt{n} < +\infty,$$

we have, for some $\alpha > 0$,

$$\lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\min(x_n, y_n))\right) = \lim_{n\to+\infty} \sqrt{n}\ s\left(\exp(\max(x_n, y_n))\right) = \alpha.$$

We have the following generalization.

Theorem 2. *If F satisfies Assumptions (Ga) and (Gb), we have*

$$X^{(n)} - F^{-1}\left(1-e^{-n}\right) \rightsquigarrow \mathcal{N}(0, \alpha^2)$$

Comments. A firm look at the results shows that for any $F \in \mathcal{D}$, we found the direct asymptotic law of $X^{(n)}$ or that of a function of $X^{(n)}$, mainly $\log X^{(n)}$. For example, Point (d) of Theorem 1 cannot be applied when X follows a lognormal law but can be applied to $\exp(X)$. This leads to the following rule for all any $F \in \mathbb{D}$:

(e) If $F \in D(G_\gamma)$, $\gamma \neq 0$, we apply Points (a) or (c) without any further condition.

(f) If $F \in D(G_0)$ and $\exp(X) \in D(G_\gamma)$ for some $\gamma > 0$, we apply Point (b) without any further condition.

(g) If $F \in D(G_0)$ and $s(u) \to 0$ as $u \to 0$. If (Ha) and (Hb) holds, we conclude by applying Point (d). If not (as it is for a lognormal law), we search whether $X_1 = \exp(X) \in D(G_\gamma)$ for some $\gamma > 0$ or $X_1 = \exp(X)$ fulfills (Ha) and (Hb). If yes, we conclude by Point (b) or by Point (d). If not, we consider $X_2 = \exp(X_1)$, and we continue until we reach $X_p = \exp(X_{p-1}) \in D(G_\gamma)$ for some $\gamma > 0$ or $X_p = \exp(X_{p-1})$ for some $p \geq 1$.

3. Examples and applications

Let us begin to explain how to apply the results for $\gamma = 0$. Generally, we may find the function $s(u)$ of $u \in]0,1[$ by from the π-variation formula

$$\forall\ \lambda > 0,\ \frac{F^{-1}(1-\lambda u) - F^{-1}(1-u)}{s(u)} \to -\log \lambda \ as\ u \to 0.$$

Another method concerns the special case where F is differentiable on left neighborhood of *uep(F)*. It is proved in Lo (1986) that if $u\left(F^{-1}(1-u)\right)'$ is slowly varying at zero, we have for some $u_0 \in]0,1[$,

$$s(u) = -u\left(F^{-1}(1-u)\right)'\ for\ u \in]0, u_0[.$$

Checking hypothesis (Ha) and (Hb) can be done with the function $s(u)$ of $u \in]0,1[$, found as explained above.

Here are some specific examples. The details for each case is given in the Appendix (Section 6, 15). We begin for light tails :

I - $F \in D(G_0)$.

(1) X **follows an exponential law** $\mathcal{E}(\lambda)$**,** $\lambda > 0$. By Point (b) of Theorem 1,

$$\frac{X^{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, \lambda^{-2}).$$

(2) X **follows a standard normal law** $\mathcal{N}(0,1)$. By Point (d) of Theorem 1,

$$X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0, 1/2).$$

(3) X **follows a Rayleigh law of parameter** $\rho > 0$, with *cdf*

$$1 - F(x) = \exp(-\rho x^2), \ x \geq 0.$$

By Point (d) of Theorem 1, we have

$$X^{(n)} - \left(\frac{n}{\rho}\right)^{1/2} \rightsquigarrow \mathcal{N}(0, \rho^{-1}/4).$$

(4) X **follows the logistic law**, with *cdf*

$$F(x) = \frac{1}{1 + e^{-x}}, \ x \in \mathbb{R}.$$

By Point (b) of Theorem 1, we have

$$\frac{X^{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, 1).$$

(5) $X > 0$ **follows a standard lognormal law, that is** $\log X$ **follows a standard normal law**. We have

$$\log X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0, 1/2).$$

(6) $X > 0$ **a follows a Gumbel law** with *cdf*

$$F(x) = \exp\left(-e^{-x}\right), \ x \in \mathbb{R}.$$

By Point (b) of Theorem 1, we have

$$\frac{X^{(n)}-n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0,1).$$

II - $F \in D(G_\gamma)$**,** $\gamma > 0$.

(7) X **follows a log-logistic law of parameter** $p > 0$, with *cfd*

$$F(x) = \frac{x^p}{1+x^p}, \ x \geq 0.$$

By Point (a) of Theorem 1,

$$\left(e^{-n/p}X^{(n)}\right)^{-1/2} \rightsquigarrow LN(0,p^2).$$

(8) X **follows a sing-Maddala law of parameters** $a > 0$**,** $b > 0$ **and** $c > 0$, with *cdf*

$$1-F(x) = \left(\frac{1}{1+ax^b}\right)^c, \ x \geq 0.$$

By Point (a), we have

$$\left(a^{1/b}\exp(-n/(bc))X^{(n)}\right)^{1/\sqrt{n}} \rightsquigarrow LN(0,(bc)^{-2}).$$

4. Proofs

(I) - Proof of Theorem 1.

We begin by describing the main tools which are based on following results of Records theory. Suppose that $\{T, T_j > 0, \ 1 \leq j \leq k\}$ are $(k+1)$ non-negative real-valued and *iid* random variables and define

$$X_0 = 0, \ T_j = X_j - X_{j-1}, \ 1 \leq j \leq k.$$

It is clear that if $T \sim \mathcal{E}(\lambda)$, $\lambda > 0$, then the absolutely continuous *pdf* of $T = (T_1, \cdots, T_k)^t$ is given by

$$f_T(t_1, \cdots, t_k) = \lambda^k e^{-\lambda t_k} \ 1_{(0 \leq t_1 \leq \cdots \leq t_k)}. \tag{4.1}$$

Suppose if T_j's are independent and follow an exponential law $\mathcal{E}(\lambda)$, $\lambda > 0$, we have

$$r(x) = \frac{dF(x)/dx}{1 - F(x)} = \lambda \ and \ f(x) = \lambda e^{-\lambda x}, \ x \geq 0.$$

As stated in page 3 in Ahnsanullah (1995), the joint distribution of the k first records values $(T^{(1)}, \cdots, T^{(k)})$ of the sequence $(T_n)_{n \geq 1}$ is the one given in Formula (4.1). As a consequence, we have

Fact 1. *If the the T_j's are independent and follow an exponential law $\mathcal{E}(\lambda)$, the k-th record value, $k \geq 1$, has the same law as the sum of k independent $\mathcal{E}(\lambda)$-random variables $E_{1,k}$, $\cdots$, $E_{k,k}$, i.e.*

$$T^{(k)} =_d E_{1,k} + \cdots + E_{k,k},$$

where $=_d$ stands for the equality in distribution. By the Renyi's representation, we can represented the random variable X of *cdf* F by a standard exponential random variable E

$$X =_d F^{-1}\left(1 - e^{-E}\right).$$

It comes that, by considering *iid* sequence $(X_n)_{n \geq 1}$ and $(E_n)_{n \geq 1}$ from X and E and by denoting the two n-th records valued $X^{(n)}$ and $E^{(n)}$ from the two sequences respectively, we have the following representations

$$X^{(n)} =_d F^{-1}\left(1 - e^{-E^{(n)}}\right),$$

where

$$S_{(n)} = E^{(n)} = E_{1,n} + \cdots + E_{n,n}.$$

In the sequel, we can and do use the equality : $X^{(n)} = F^{-1}\left(1 - e^{-S_{(n)}}\right)$. Let us apply the representations by using the simple central limit theorem

$$\frac{S_{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, 1) \ as \ n \to +\infty.$$

In the sequel, any unspecified limit is meant as $n \to +\infty$.

Let us suppose $X \in D(G_{1/\gamma})$. If $X \geq 0$, we will consider $Y = \log X$ of *cdf* G defined by $G(x) = F(e^x)$, $x \in \mathbb{R}$. Let us prove the theorem.

(a) - Asymptotic law of $X^{(n)}$ for $\gamma > 0$. We recall that $V_n = e^{-S_{(n)}}$ and $v_n = e^{-n}$, $n \geq 1$. By Representation (1.4), we have

$$F^{-1}\left(1 - e^{-S_{(n)}}\right) = (1 + a(V_n))V_n^{-\gamma} \exp\left(\int_{V_n}^{1} \frac{b(t)}{t}\, dt\right), \ n \geq 1$$

and

$$F^{-1}\left(1 - e^{-n}\right) = (1 + a(v_n))v_n^{-\gamma} \exp\left(\int_{v_n}^{1} \frac{b(t)}{t}\, dt\right), \ n \geq 1.$$

We get that $V_n \to_{\mathbb{P}} 0$, $(1 + a(V_n))/(1 + a(v_n)) \equiv 1 + p_n \to_{\mathbb{P}} 1$. We get

$$\log\left(\frac{X^{(n)}}{F^{-1}\left(1 - e^{-n}\right)}\right) = p_n(1 + o_{\mathbb{P}}(1)) - \gamma(S_{(n)} - n) + \int_{v_n}^{V_n} \frac{b(t)}{t}\, dt.$$

We have

$$\left|\int_{v_n}^{1} \frac{b(t)}{t}\, dt\right| \leq \left(\sup_{0 \leq t \leq (v_n \vee V_n)} |b(t)|\right)|S_{(n)} - n|.$$

By combining the two later formulae, we have

$$n^{-1/2} \log\left(\frac{X^{(n)}}{F^{-1}\left(1 - e^{-n}\right)}\right) \rightsquigarrow \mathcal{N}(0, \gamma^2).$$

(b) - Asymptotic law of $Y^{(n)}$ for $\gamma > 0$. From the previous theorem, it is immediate for the following result. It is clear the $G^{-1} = \log F^{-1}$. So, the previous theorem implies

$$n^{-1/2}\left(Y^{(n)} - G^{-1}\left(1 - e^{-n}\right)\right) \rightsquigarrow \mathcal{N}(0, \gamma^2).$$

Here, it is clear that $Y \in D(G_0)$ and $R(x, G) \to \gamma$ as $x \to uep(G)$. Hence this result says that

$$n^{-1/2}\left(X^{(n)} - F^{-1}\left(1 - e^{-n}\right)\right) \rightsquigarrow \mathcal{N}(0, \gamma^2),$$

if $F \in D(G_0)$ and $R(x,F) \to \gamma$ as $x \to uep(F)$.

(c) - Asymptotic law of $Y^{(n)}$ **for** $\gamma < 0$. We have $\mathbb{P}(X = uep(F)) = 0$. By using Representation (1.5), we may and do prove this point exactly as for Point (a).

(d) - Asymptotic law of $Y^{(n)}$ **for** $\gamma = 0$. We did not have yet the general law. Let us learn for a no-trivial example.

(A) - $X \sim \mathcal{N}(0,1)$. Let us recall the expansion of the tail of F as follows

$$F^{-1}(1-s) = (2\log(1/s))^{1/2} - \frac{\log 4\pi + \log\log(1/s)}{2(2\log(1/s))^{1/2}} + O((\log\log(1/s)^2(\log 1/s)^{-1/2})). \tag{4.2}$$

We have

$$\begin{aligned} X^{(n)} &= (2S_{(n)})^{1/2} - \frac{\log 4\pi + \log S_{(n)}}{2(2S_{(n)}))^{1/2}} + O(S_{(n)}^{-1/2}\log S_{(n)}) \\ &= (2S_{(n)})^{1/2}\left(1 - \frac{\log 4\pi + \log\log(1/s)}{4S_{(n)}} + O(S_{(n)}(\log S_{(n)})\right) \\ &= (2S_{(n)})^{1/2}(1+\varepsilon_n). \end{aligned}$$

We have that $\varepsilon_n = O_{\mathbb{P}}(n^{-1})$. Let us use the mean value theorem to get

$$S_{(n)}^{1/2} - n^{1/2} = \frac{1}{2}\frac{S_{(n)} - n}{\sqrt{n}}(n/\zeta_n)^{1/2},$$

with $n \wedge S_{(n)} < \zeta_n < n \vee S_{(n)}$ and next, by the weak law of large numbers, $2(S_{(n)}^{1/2} - n^{1/2}) \rightsquigarrow \mathcal{N}(0,1)$. By plugging this in the later formula, we get

$$\left(X^{(n)} - (2n)^{1/2}\right) - \left(\sqrt{2}(S_{(n)})^{1/2} - n^{1/2})\right) = O_{\mathbb{P}}(\varepsilon_n) + O_{\mathbb{P}}(n^{-1/2}). \tag{4.3}$$

We conclude that

$$X^{(n)} - (2n)^{1/2} \rightsquigarrow \mathcal{N}(0, 1/2).$$

By putting

$$b_n = (2n)^{1/2} - \frac{\log 4\pi + \log n}{2(2n)^{1/2}}$$

we also have

$$X^{(n)} - \left((2n)^{1/2} - \frac{\log 4\pi + \log n}{2(2n)^{1/2}} \right) \rightsquigarrow \mathcal{N}(0, 1/2)$$

(B) - General proof. It known that $s(u) \sim R(F^{-1}(1-u), F)$ and so, $s(u) \to 0$ as $u \to 0$, By representation (1.6) of Proposition 2 and Hypothesis (Ha) together lead to

$$\begin{aligned} X^{(n)} - F^{-1}(1 - e^n) &= s(V_n) - s(v_n) + \int_{v_n}^{V_n} \frac{s(u)}{u} \, du \\ &= o_{\mathbb{P}}(1) + s(V_n) - s(v_n) - (1 + o_{\mathbb{P}}(1))s(v_n)(S_{(n)} - n). \end{aligned}$$

From there, the conclusion is immediate. ■

(II) - Proof of Theorem 2. We have $g(x) = F^{-1}(1 - e^{-x})$ $g'(x) = S(x)$, $x \in]lep(F), uep(F)[$. The mean value theorem gives, for

$$X^{(n)} - F^{-1}(1 - e^{-n}) = \frac{S_{(n)} - n}{\sqrt{n}} \left(\sqrt{n} S(\exp(-\zeta_n)) \right), \tag{4.4}$$

where

$$\zeta_n \in] \min(n, S_{(n)}), \ \max(n, S_{(n)})[.$$

From there, the conclusion is direct. ■

5. Conclusion

After the statements of the asymptotic laws of the strong record values from *iid* random variables and after some examples have been given, it should be interesting to a review of such asymptotic laws for as much as possible *cdf*'s $F \in \mathcal{D}$.

6. Appendix

Let us give the details concerning the results listed in Section 3.

(1) X **follows an exponential law** $\mathcal{E}(\lambda)$**,** $\lambda > 0$. We have $exp(X) \in D(G_{-1})$ and $F^{-1}(1-e^{-n}) = n$. We apply Point (b) to conclude.

$$\frac{X^{(n)} - n}{\sqrt{n}} \rightsquigarrow \mathcal{N}(0, \lambda^{-}2).$$

(2) X **follows a standard normal law** $\mathcal{N}(0,1)$. The result of this point is justified by Formula 4.3, page 13.

(3) X **a follows Rayleigh law of parameter** $\rho > 0$. We have

$$F^{-1}(1-u) = \left(-\frac{1}{\rho}\log u\right)^{1/2}, \ u \in]0,1[$$

and

$$s(u) = -u\left(F^{-1}(1-u)\right)' = \frac{1}{2\rho(-(1/\rho)logu)^{1/2}} \to 0 \ as \ u \to 0.$$

Furthermore, $s(u)$ is decreasing in $u \in]0,1[$ and $s(V_n)/s(v_n) \to 0$ as $n \to +\infty$. Finally,

$$\sqrt{n}s(v_n) \to \rho^{-1/2}/2.$$

We conclude the case by applying Point (d) of Theorem 1.

(4) X **a follows the logistic law**. It is immediate that $\exp(X) \in D(G_{-1})$ and we have

$$F^{-1}(1-u) = \log(u/(1-u)), \ u \in]0,1[.$$

We conclude with Point (b) of Theorem 1.

(5) $X > 0$ **a follows a standard lognormal law, that is** $\log X$ **follows a standard normal law**.

Since $\log X^{(n)}$ has the same law as the n-th record $Z^{(n)}$ from *iid* $\mathcal{N}(0,1)$ random variables. So we have

$$\log X^{(n)} - (2n)^{1/2} \to \mathcal{N}(0, 1/2).$$

(6) $X > 0$ **a follows a Gumbel law**. We have

$$F^{-1}(1-u) = -\log\log(1/(1-u)),\ u \in]0,1[$$

and for any $\lambda > 0$.

$$F^{-1}(1-\lambda u) - F^{-1}(1-u) = \log(\lambda(1+o(1))) \to \log\lambda \ as\ u \to 0.$$

So, $\exp(X) \in D(G_{-1})$. From there, an application of Point (b) of Theorem 1 closes the case.

(7) X **follows a log-logistic law of parameter** $p > 0$, with *cfd*

$$F(x) = \frac{x^p}{1+x^p},\ x \geq 0.$$

We have

$$F^{-1}(1-u) = u^{-1/p}(1-u)^{1/p}, u \in]0,1[.$$

By Point (a) of Theorem 1,

$$\left(e^{-n/p}X^{(n)}\right)^{1/\sqrt{n}} \rightsquigarrow LN(0,p^2).$$

(8) X **follows a sing-Maddala law of parameters** $a > 0$**,** $b > 0$ **and** $c > 0$. We have

$$1 - F(x) = x^{-bc}(x^{-b}+a)^{-c} \equiv x^{-bc}L(x),\ x \geq 0,$$

and L is a slowly varying function at $+\infty$. So $F \in G_{1/(bc)}$. Applying of Point (a) of Theorem, when combined with

$$F^{-1}(1-u) = a^{-1/b}u^{-1/(bc)}(1-u^{1/c})^{1/b},\ u \in]0,1[,$$

and with,

$$F^{-1}(1-e^{-n}) = a^{-1/b}e^{n/(bc)}(1-e^{-n/c})^{1/b},$$

for $n \geq 1$, closes the case.

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