Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

PROJECTION METHODS FOR SOLVING SPLIT EQUILIBRIUM PROBLEMS

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(Communicated by the associate editor name)

Abstract. The paper considers a split inverse problem involving component equilibrium problems in Hilbert spaces. This problem therefore is called the split equilibrium problem (SEP). It is known that almost solution methods for solving problem (SEP) are designed from two fundamental methods as the proximal point method and the extended extragradient method (or the two-step proximal-like method). Unlike previous results, in this paper we introduce a new algorithm, which is only based on the projection method, for finding solution approximations of problem (SEP), and then establish that the resulting algorithm is weakly convergent under mild conditions. Several of numerical results are reported to illustrate the convergence of the proposed algorithm and also to compare with others.

1. **Introduction.** The split feasibility problem [6] consists of finding a point in a closed convex subset of a space such that its image under a bounded linear operator belongs to a closed convex subset of another space. This problem has received a lot of attention because of its applications in signal processing, specifically in phase retrieval and other image restoration problems, see, e.g., [26, 37]. After that, it was found that the split feasibility problem can be used to model the intensity-modulated radiation therapy [8], and many other fields [3, 4, 5]. That is also the reason to explain why in recent years many split-like problems have been widely and intensively studied, for instance, the split fixed point problem, the split optimization problem and the split variational inequality problem [7, 30] and others [9, 32, 33, 40]. Mathematically, these problems can be modelled in a common form, and so-called the split inverse problem (SIP), see in [7, Sect. 2], in which there are a bounded linear operator A from a space X to another space Y and two inverse problems IP1 and IP2 installed in X and Y, respectively. More precisely, the problem (SIP) is of the form,

$$\begin{cases} \text{find a point } \ x^* \in X \text{ that solves IP1} \\ \text{such that} \\ \text{the point } \ y^* = Ax^* \in Y \text{ solves IP2.} \end{cases} \tag{SIP}$$

2010 *Mathematics Subject Classification.* Primary: 65K10, 65K15; Secondary: 90C33.
Key words and phrases. Split equilibrium problem, Split inverse problem, Projection method, Diagonal subgradient method.
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Based on this general model, we can consider various types of split problems, even extend them to split equality-like problems. Recall that the equilibrium problem [2, 10, 34] for a bifunction $f : C \times C \to \Re$ is to find a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C, \tag{EP}$$

where C is a nonempty closed convex subset of a real Hilbert space H. Let us denote by $EP(f, C)$ the solution set of the problem (EP). It was well known that problem (EP) unifies in a simple form many mathematical models such as the variational inequalities, the fixed point problems, the optimization problems and the Nash equilibrium problems, see, e.g., [2, 19, 22, 23, 24, 25, 34]. It is here natural in this framework to study problem (SIP) when IP1 and IP2 are equilibrium problems to get the so-called split equilibrium problem (SEP). The problem of this form has also been considered recently in [18, 21, 30]. More precisely, the problem (SEP) is stated as follows:

Problem (SEP): *Let $H_1, \ H_2$ be two real Hilbert spaces and $C, \ Q$ be two nonempty closed convex subsets of $H_1, \ H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $f : C \times C \to \Re$ and $F : Q \times Q \to \Re$ be two bifunctions with $f(x, x) = 0$ for all $x \in C$ and $F(u, u) = 0$ for all $u \in Q$. The problem (SEP) is:*

$$\begin{cases} \textit{Find } x^* \in C \textit{ such that } f(x^*, y) \geq 0, \ \forall y \in C, \\ \textit{and } u^* = Ax^* \in Q \textit{ solves } F(u^*, v) \geq 0, \ \forall v \in Q. \end{cases} \tag{SEP}$$

Let Ω denote the solution set of problem (SEP), i.e.,

$$\Omega = \{x^* \in C : f(x^*, y) \geq 0 \text{ and } F(Ax^*, v) \geq 0, \ \forall y \in C, \ \forall v \in Q\}.$$

Several methods for solving problem (SEP) can be found, for instance, in [12, 13, 18, 20, 28, 30]. As far as we know, almost solution methods for solving problem (SEP) are based on the proximal point method [29] which consists of computing the resolvents T_r^f and T_s^F of bifunctions $f, \ F$ with some $r, \ s > 0$. Recall that the resolvent [10] of a bifunction $f : C \times C \to \Re$ with some $r > 0$ is defined by

$$T_r^f(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}. \tag{1}$$

Recently, the author of [20] has introduced the extragradient-proximal method [20, Corollary 1], for solving problem (SEP), which combines three methods including the proximal point method [29], the extended extragradient method [16, 35] and the projection method. Also, recall here that the extended extragradient method [16, 35] involves the computations of the following two optimization programs, for each $x \in C$,

$$\begin{cases} y = \arg\min\{\lambda f(x, t) + \frac{1}{2}||x - t||^2 : t \in C\}, \\ z = \arg\min\{\lambda f(y, t) + \frac{1}{2}||x - t||^2 : t \in C\}, \end{cases} \tag{2}$$

where some $\lambda > 0$. It seems that the extended extragradient method (2) can be easier to compute numerically than the proximal-point method, that is $T_r^f(x)$, which comes from the nonlinear inequality in (1). However, the solving of two optimization programs in (2) can be still costly if the bifunction f and the feasible set C have complex structures. Very recently, the author of [21] has presented a new algorithm (see, [21, Algorithm 3.1]) for solving problem (SEP), namely the projected subgradient - proximal method (PSPM). More precisely, the PSPM is designed as follows:

Algorithm (PSPM)

Initialization: Choose $x_0 \in C$ and the parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{r_n\}$, $\{\mu_n\}$ such that

(i) $\rho_n \geq \rho > 0$, $\beta_n > 0$, $\epsilon_n \geq 0$, $r_n \geq r > 0$.

(ii) $\sum_{n\geq 1} \frac{\beta_n}{\rho_n} = +\infty, \qquad \sum_{n\geq 1} \frac{\beta_n \epsilon_n}{\rho_n} < +\infty, \qquad \sum_{n\geq 1} \beta_n^2 < +\infty.$

(iii) $0 < a \leq \mu_n \leq b < \frac{2}{||A||^2}$.

Iterative Steps: Assume that $x_n \in C$ is known, calculate x_{n+1} as follows:

Step 1. Select $w_n \in \partial_{\epsilon_n} f(x_n, .)(x_n)$, and compute

$$\gamma_n = \max\{\rho_n, ||w_n||\}, \qquad \alpha_n = \frac{\beta_n}{\gamma_n}, \qquad y_n = P_C(x_n - \alpha_n w_n).$$

Step 2. Compute $x_{n+1} = P_C\left(y_n - \mu_n A^*(I - T_{r_n}^F)Ay_n\right)$.

A modification of PSPM was also introduced in [21, Algorithm 4.1] where the prior knowledge of the norm of operator A is not necessary. As be seen, the PSPM is constructed around the two methods, namely the projection method and the proximal point method [29] (i.e., using the resolvent mapping $T_{r_n}^F$ of bifunction F). Finding a value of resolvent mapping in general is not easy. Then, the introduction of a computable and effective algorithm is necessary.

***Our concern now is the following**: Can we construct an algorithm for solving problem (SEP) which only uses the projection method?*

In this paper, as a continuity of the results in [20, 21], we introduce a different algorithm for approximating solutions of problem (SEP) to answer the aforementioned question. Unlike the existing results, we only use the projection methods to design the algorithm. Theorem of weak convergence is proved under mild conditions. For further purpose, we consider some simple examples to demonstrate that several considered conditions are necessary in the formulation of theorem of convergence. The resulting algorithm is also extended to solve other related form-like problems. Finally, we perform several experiments to illustrate the numerical behavior of the new algorithm and aslo to compare it with others. The analyses in this paper are based on the ones in the recent work [36]. In this direction, a special case of problem (SEP) has been studied by the authors in [1]. A generalization of problem (SEP) with fixed point problems and the methods of proximal-extragradient form can be found in [14].

An outline of this paper is as follows: In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with proposing the algorithm and analyzing its convergence. Some further remarks are presented in Sect. 4 to justify the introduction of the assumptions in the convergence theorem. Sect. 5 introduces an extension of the resulting algorithm to the split common equilibrium problem. In Sect. 6 we perform several numerical experiments to illustrate the computational efficiency of the proposed algorithm and also to compare it with others.

2. **Preliminaries.** Let C be a nonempty closed convex subset of a real Hilbert space H. The metric projection $P_C : H \to C$ is defined by

$$P_C(x) = \arg\min \{\|y - x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C(x)$ exists and is unique. From the definition of the metric projection, it is easy to show that P_C has the following property.

Lemma 2.1. [17] (i) $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in H.$

(ii) $\|x - P_C(y)\|^2 + \|P_C(y) - y\|^2 \leq \|x - y\|^2, \forall x \in C, y \in H.$

(iii) $z = P_C(x) \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$

Now, we recall some concepts of monotonicity of a bifunction, see, e.g., [2, 34].

Definition 2.2. A bifunction $f : C \times C \to \Re$ is said to be:

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma||x - y||^2, \ \forall x, y \in C;$$

(ii) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \ \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0, \ \forall x, y \in C.$$

(iv) strongly pseudomonotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq -\gamma||x - y||^2, \ \forall x, y \in C.$$

From the above definitions, it is clear that the following implications hold,

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii) and (i)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(iii)}.$$

The converses in general are not true. Recall that a function $\varphi : C \to \Re$ is said to be convex on C if for all $x, y \in C$ and $t \in [0, 1]$,

$$\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y).$$

The subdifferential of φ at $x \in C$ is defined by

$$\partial\varphi(x) = \{w \in H : \varphi(y) - \varphi(x) \geq \langle w, y - x\rangle, \ \forall y \in C\}.$$

An enlargement of the subdifferential is the ϵ-subdifferential. The ϵ-subdifferential of φ at $x \in C$ is defined by

$$\partial_\epsilon\varphi(x) = \{w \in H : \varphi(y) - \varphi(x) + \epsilon \geq \langle w, y - x\rangle, \ \forall y \in C\}.$$

It is clear that the 0-subdifferential coincides with the subdifferential. Let $f : C \times C \to \Re$ be a bifunction. Throughout this paper, $\partial_\epsilon f(x, .)(x)$ is called the ϵ-diagonal subdifferential of f at $x \in C$.

We need the following technical lemma to prove the convergence of the proposed algorithms.

Lemma 2.3. [39] *Let* $\{\nu_n\}$ *and* $\{\delta_n\}$ *be two sequences of positive real numbers such that*

$$\nu_{n+1} \leq \nu_n + \delta_n, \ \forall n \geq 1,$$

with $\sum_{n\geq 1} \delta_n < +\infty$. *Then the sequence* $\{\nu_n\}$ *is convergent.*

3. **Algorithm and convergence.** In this section, we introduce a new algorithm for approximating solutions of problem (SEP). For designing our algorithm, throughout the paper, we take four non-negative parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, and $\{\mu_n\}$ satisfying the following conditions.

C1. $\rho_n \geq \rho > 0$, $\epsilon_n \geq 0$, $\beta_n > 0$.

C2. $\sum\limits_{n\geq 1} \frac{\beta_n}{\rho_n} = +\infty, \qquad \sum\limits_{n\geq 1} \frac{\beta_n \epsilon_n}{\rho_n} < +\infty, \qquad \sum\limits_{n\geq 1} \beta_n^2 < +\infty.$

C3. $0 < a \leq \mu_n \leq \frac{1}{||A||^2}$.

The following is the algorithm in details.

Algorithm 1 (Projection Method for SEPs)**.** .

Initialization: Choose $x_0 \in C$ and parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{\mu_n\}$ such that conditions C1-C3 above hold.

Iterative Steps: Assume that $x_n \in C$ is known, calculate x_{n+1} as follows:

Step 1. Select $w_n \in \partial_{\epsilon_n} F(u_n, .)(u_n)$ where $u_n = P_Q(Ax_n)$, and compute

$$\gamma_n = \frac{\beta_n}{\max\{\rho_n, ||w_n||\}}, \qquad y_n = P_Q(u_n - \gamma_n w_n).$$

Step 2. Compute $z_n = P_C\left(x_n + \mu_n A^*(y_n - Ax_n)\right)$.

Step 3. Select $g_n \in \partial_{\epsilon_n} f(z_n, .)(z_n)$ and compute

$$\alpha_n = \frac{\beta_n}{\max\{\rho_n, ||g_n||\}}, \qquad x_{n+1} = P_C(z_n - \alpha_n g_n).$$

Set $n := n + 1$ and go back to **Step 1**.

Remark 1. Remark that since $\rho_n \geq \rho > 0$, Algorithm 1 is well defined. In view of the existing methods in [9, 12, 13, 14, 20, 28, 30], we see that they are almost designed in combining two methods: the proximal point method (that is to compute the resolvent of a bifunction [29]) or the extended extragradient method (or the two-step proximal-like method [16, 35]). Algorithm 1 is close to the methods in [21]. However, it only uses the projection method to design it and without the resolvent mapping T_r^F of F as in [21]. As mentioned above, the using of resolvent mapping can be time-consuming in numerical computation when the bifunctions and the feasible sets have complicated structures.

Remark 2. From the condition $\sum\limits_{n=1}^{\infty} \beta_n^2 < +\infty$ in (C2), we see that $\lim\limits_{n\to\infty} \beta_n = 0$. This implies that the sequences of stepsizes $\{\gamma_n\}$ and $\{\alpha_n\}$ in Algorithm 1 are decreasing. In general this strategy is not good. However, this assumption allows the algorithm to work without imposing the Lipschitz-type condition on the bifunction. So doing, the stepsizes are suitably updated at each iteration and are independent on the Lipschitz-type constants.

In order to establish the convergence of Algorithm 1, we assume that the bifunction $f : C \times C \to \Re$ satisfies the following conditions.

A1. f is pseudomonotone and $f(x,x) = 0, \ \forall x \in C$;

A2. $f(x,.)$ is convex and lower semicontinuous on C and $f(.,y)$ is weakly upper semicontinuous on C;

A3. The ϵ-diagonal subdifferential of f is bounded on each bounded subset of C;

A4. f satisfies the following paramonotone condition

$$x \in EP(f,C), \ y \in C, \ f(y,x) = 0 \Longrightarrow y \in EP(f,C).$$

In addition, the bifunction $F : Q \times Q \to \Re$ is also assumed to satisfy the properties A1 - A4 above, but on the feasible set Q. Several remarks on these assumptions will be presented in the next section where it is seen that paramonotone condition A4 is necessary to establish the convergence of the algorithm. Under conditions A1 and A2, the two sets $EP(f,C)$ and $EP(F,Q)$ are closed and convex. Thus, since A is linear, the solution set Ω of problem (SEP) is also closed and convex. In this paper, Ω is assumed to be nonempty, and so the projection $P_\Omega(u)$ is well defined for each point $u \in H_1$. To investigate the asymptotic behavior of the sequence $\{x_n\}$ generated by Algorithm 1, we need the following lemmas.

Lemma 3.1. *Let $x^* \in EP(f,C)$. Then we have the following estimate for each $n \geq 0$,*

$$||x_{n+1} - x^*||^2 \leq ||z_n - x^*||^2 + 2\alpha_n f(z_n, x^*) - ||x_{n+1} - z_n||^2 + \delta_n,$$

where $\delta_n = \frac{2\beta_n\epsilon_n}{\rho_n} + 2\beta_n^2$.

Proof. See, e.g., [21, inequality (8)]. □

As in Lemma 3.1 but with $Ax^* \in EP(F,Q)$, from the definition of y_n, we also have the following estimate for each $n \geq 0$,

$$||y_n - Ax^*||^2 \leq ||u_n - Ax^*||^2 + 2\gamma_n F(u_n, Ax^*) - ||y_n - u_n||^2 + \delta_n,$$

where δ_n is defined as in Lemma 3.1. Thus

$$||y_n - Ax^*||^2 \leq ||u_n - Ax^*||^2 + 2\gamma_n F(u_n, Ax^*) + \delta_n. \tag{3}$$

Lemma 3.2. *Let $x^* \in \Omega$. Then the following inequality holds for each $n \geq 0$,*

$$\begin{aligned} ||x_{n+1} - x^*||^2 \quad \leq \quad & ||x_n - x^*||^2 - \mu_n ||u_n - Ax_n||^2 - ||x_{n+1} - z_n||^2 \\ & + 2\mu_n\gamma_n F(u_n, Ax^*) + 2\alpha_n f(z_n, x^*) + (1+\mu_n)\delta_n. \end{aligned}$$

Proof. From the definition of z_n and the nonexpansiveness of P_C, we obtain

$$\begin{aligned} ||z_n - x^*||^2 &= ||P_C\left(x_n + \mu_n A^*(y_n - Ax_n)\right) - P_C(x^*)||^2 \\ &\leq ||x_n + \mu_n A^*(y_n - Ax_n) - x^*||^2 \\ &= ||x_n - x^*||^2 + \mu_n^2 ||A^*(y_n - Ax_n)||^2 + 2\mu_n \left\langle x_n - x^*, A^*(y_n - Ax_n)\right\rangle \\ &\leq ||x_n - x^*||^2 + \mu_n^2 ||A||^2 ||y_n - Ax_n||^2 + 2\mu_n \left\langle A(x_n - x^*), y_n - Ax_n\right\rangle. \end{aligned} \tag{4}$$

Now, we estimate the term $\langle A(x_n - x^*), y_n - Ax_n\rangle$ in inequality (4). Since P_Q is firmly nonexpansive, we obtain

$$\begin{aligned} 2||u_n - Ax^*||^2 &= 2||P_Q(Ax_n) - P_Q(Ax^*)||^2 \\ &\leq 2\langle P_Q(Ax_n) - P_Q(Ax^*), Ax_n - Ax^*\rangle \\ &= 2\langle u_n - Ax^*, Ax_n - Ax^*\rangle \\ &= ||u_n - Ax^*||^2 + ||Ax_n - Ax^*||^2 - ||u_n - Ax_n||^2. \end{aligned}$$

Thus

$$||u_n - Ax^*||^2 \leq ||Ax_n - Ax^*||^2 - ||u_n - Ax_n||^2. \tag{5}$$

Combining relations (3) and (5) we obtain

$$||y_n - Ax^*||^2 \leq ||Ax_n - Ax^*||^2 - ||u_n - Ax_n||^2 + 2\gamma_n F(u_n, Ax^*) + \delta_n, \tag{6}$$

which, together with the following equality

$$2\langle A(x_n - x^*), y_n - Ax_n\rangle = ||y_n - Ax^*||^2 - ||Ax_n - Ax^*||^2 - ||y_n - Ax_n||^2,$$

implies that

$$2\langle A(x_n - x^*), y_n - Ax_n\rangle \leq -||u_n - Ax_n||^2 - ||y_n - Ax_n||^2 + 2\gamma_n F(u_n, Ax^*) + \delta_n. \tag{7}$$

Combining relations (4) and (7), we get

$$\begin{aligned} ||z_n - x^*||^2 &\leq ||x_n - x^*||^2 - \mu_n(1 - \mu_n||A||^2)||y_n - Ax_n||^2 - \mu_n||u_n - Ax_n||^2 \\ &\quad + 2\mu_n\gamma_n F(u_n, Ax^*) + \mu_n\delta_n \\ &\leq ||x_n - x^*||^2 - \mu_n||u_n - Ax_n||^2 + 2\mu_n\gamma_n F(u_n, Ax^*) + \mu_n\delta_n, \end{aligned} \tag{8}$$

where the last inequality follows from assumption C3 that $\mu_n(1 - \mu_n||A||^2) \geq 0$. This together with Lemma 3.1 implies the desired conclusion. Lemma 3.2 is proved. □

Lemma 3.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 1. Then the following properties are satisfied.*

(i) *The sequence $\left\{||x_n - x^*||^2\right\}$ is convergent for each $x^* \in \Omega$, and the sequence $\{x_n\}$ is bounded.*

(ii) $\lim_{n\to\infty} ||x_{n+1} - z_n||^2 = \lim_{n\to\infty} ||u_n - Ax_n||^2 = 0$, *and the sequences $\{z_n\}$, $\{u_n\}$ are bounded.*

(iii) $\lim_{n\to\infty} \sup f(x_n, x^*) = \lim_{n\to\infty} \sup F(u_n, Ax^*) = 0$ *for each $x^* \in \Omega$.*

Proof. (i) Since $x^* \in EP(f, C)$ and $x_n \in C$, we have $f(x^*, x_n) \geq 0$. Then $f(x_n, x^*) \leq 0$ by the pseudomonotonicity of f. Similarly, from $u_n \in Q$ and $Ax^* \in EP(F, Q)$, we also have $F(u_n, Ax^*) \leq 0$. These together with Lemma 3.2, $\mu_n > 0$, $\alpha_n > 0$, $\gamma_n > 0$ imply that

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + (1 + \mu_n)\delta_n. \tag{9}$$

Using Lemma 2.3 and the fact that $\sum_{n\geq 1}(1 + \mu_n)\delta_n < +\infty$, it follows that the sequence $\left\{||x_n - x^*||^2\right\}$ converges and thus that $\{x_n\}$ is bounded.

(ii) For the sake of simplicity, we set

$$M_n = \mu_n||u_n - Ax_n||^2 + ||x_{n+1} - z_n||^2 \geq 0,$$

$$N_n = -2\mu_n\gamma_n F(u_n, Ax^*) - 2\alpha_n f(z_n, x^*) \geq 0.$$

Thus, the inequality in Lemma 3.2 can be shortly rewritten as

$$M_n + N_n \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2 + (1 + \mu_n)\delta_n.$$

Let $N \geq 1$ be a fixed integer number. Summing up these inequalities for $n = 1, 2, \ldots, N$, we obtain

$$0 \leq \sum_{n=1}^{N} M_n + \sum_{n=1}^{N} N_n \leq ||x_1 - x^*||^2 - ||x_{N+1} - x^*||^2 + \sum_{n=1}^{N}(1+\mu_n)\delta_n.$$

This is true for all $N \geq 1$. Passing to the limit in the last inequality as $N \to \infty$, and using Lemma 3.3(i) and the fact that $\sum_{n\geq 1}(1+\mu_n)\delta_n < +\infty$, we obtain

$$\text{(S1)} \quad \sum_{n=1}^{\infty} M_n < +\infty, \qquad \text{(S2)} \quad \sum_{n=1}^{\infty} N_n < +\infty.$$

From (S1) and the definition of M_n, we obtain

$$\lim_{n\to\infty} ||x_{n+1} - z_n||^2 = 0 \tag{10}$$

and $\lim_{n\to\infty} \mu_n ||u_n - Ax_n||^2 = 0$. This together with the hypothesis $\mu_n \geq a > 0$ implies that

$$\lim_{n\to\infty} ||u_n - Ax_n||^2 = 0. \tag{11}$$

Thus, from the boundedness of $\{x_n\}$ and the linearity of operator A, we also obtain that the two sequences $\{z_n\}$, $\{u_n\}$ are bounded.

(iii) From (S2), the definition of N_n, and the facts $-\mu_n\gamma_n F(u_n, Ax^*) \geq 0$ and $-2\alpha_n f(z_n, x^*) \geq 0$ for all $n \geq 0$, we obtain

$$\text{(S3)} \quad \sum_{n=1}^{\infty} \alpha_n[-f(z_n, x^*)] < +\infty, \qquad \text{(S4)} \quad \sum_{n=1}^{\infty} \mu_n\gamma_n[-F(u_n, Ax^*)] < +\infty.$$

Hence from (S4) and hypothesis C3, we can deduce that

$$\text{(S5)} \quad \sum_{n=1}^{\infty} \gamma_n[-F(u_n, Ax^*)] < +\infty.$$

On the other hand, since $\{z_n\}$ is bounded, it follows from assumption A3 that $\{g_n\}$ is also bounded. Thus, there exists $L \geq \rho > 0$ such that $||g_n|| \leq L$, and from the definition of α_n and C1, we can write

$$\alpha_n = \frac{\beta_n}{\max\{\rho_n, ||w_n||\}} = \frac{\beta_n}{\rho_n \max\{1, ||w_n||/\rho_n\}} \geq \frac{\beta_n}{\rho_n}\frac{\rho}{L}.$$

This together with (S3) and $\frac{\rho}{L} > 0$ implies that

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\rho_n}[-f(z_n, x^*)] < +\infty.$$

Consequently, under hypothesis C2 we obtain that $\lim_{n\to\infty} \inf[-f(z_n, x^*)] = 0$, i.e.,

$$\lim_{n\to\infty} \sup f(z_n, x^*) = 0.$$

Similarly, from the boundedness of $\{u_n\}$ and (S5), we also get that

$$\lim_{n\to\infty} \sup F(u_n, Ax^*) = 0.$$

This completes the proof of Lemma 3.3. □

Now, we prove the convergence of Algorithm 1.

Theorem 3.4. *The whole sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to some solution $x^\dagger$ of problem (SEP). Moreover, $x^\dagger = \lim_{n\to\infty} P_\Omega(x_n)$.*

Proof. Since $\{z_n\}$ is bounded, without loss of generality, we can assume that there exists a subsequence $\{z_m\}$ of $\{z_n\}$ converging weakly to $x^\dagger$ such that

$$\lim_{n\to\infty} \sup f(z_n, x^*) = \lim_{m\to\infty} f(z_m, x^*). \tag{12}$$

Since C is closed and convex in a Hilbert space, C is weakly closed in H_1. Thus, from $\{z_m\} \subset C$, we get that $x^\dagger \in C$. Then, it follows from the weak upper semicontinuity of $f(., x^*)$, relation (12) and Lemma 3.3(iii) that

$$f(x^\dagger, x^*) \geq \lim_{m\to\infty} \sup f(z_m, x^*) = \lim_{m\to\infty} f(z_m, x^*) = \lim_{n\to\infty} \sup f(z_n, x^*) = 0. \tag{13}$$

Since $x^* \in EP(f, C)$ and $x^\dagger \in C$, we have $f(x^*, x^\dagger) \geq 0$. Thus, from the pseudomonotonicity of f, we get that $f(x^\dagger, x^*) \leq 0$. This together with relation (13) implies that $f(x^\dagger, x^*) = 0$ and, using A4, that $x^\dagger \in EP(f, C)$.

Now we show that $Ax^\dagger \in EP(F, Q)$ and thus $x^\dagger \in \Omega$. Since $z_m \rightharpoonup x^\dagger$ and $||x_{m+1} - z_m|| \to 0$, from Lemma 3.3(ii), we also have $x_{m+1} \rightharpoonup x^\dagger$, and thus $Ax_{m+1} \rightharpoonup Ax^\dagger$. Furthermore, also from Lemma 3.3(ii), we see that $||u_{m+1} - Ax_{m+1}||^2 \to 0$ as $m \to \infty$, and thus $u_{m+1} \rightharpoonup Ax^\dagger$. The feasible set Q being weakly closed and the subsequence $\{u_{m+1}\}$ being contained in Q, we obtain that $Ax^\dagger \in Q$. Arguing as in (12) and (13) but for the bifunction F, we also obtain that $Ax^\dagger \in EP(F, Q)$.

Since $x^\dagger \in \Omega$ and Lemma 3.3(i), we can claim that the sequence $\{||x_n - x^\dagger||^2\}$ is convergent. Thus, from Lemma 3.3(ii), we also obtain the convergence of the sequence $\{||z_n - x^\dagger||^2\}$. Now, we show the whole sequence $\{z_n\}$ converges weakly to $x^\dagger$. Indeed, assume that $\bar{x}$ is a weak cluster of the sequence $\{z_n\}$ such that $\bar{x} \neq x^\dagger$, i.e., there exists a subsequence $\{z_k\}$ of $\{z_n\}$ converging weakly to $\bar{x}$. It is obvious that $\bar{x} \in \Omega$ and thus that the sequence $\{||z_n - \bar{x}||^2\}$ is convergent. We have the following equality,

$$2\left\langle z_n, \bar{x} - x^\dagger \right\rangle = ||z_n - x^\dagger||^2 - ||z_n - \bar{x}||^2 + ||\bar{x}||^2 - ||x^\dagger||^2.$$

Thus, the limit of the sequence $\{\langle z_n, \bar{x} - x^\dagger \rangle\}$ exists and is denoted by l, i.e.,

$$\lim_{n\to\infty} \left\langle z_n, \bar{x} - x^\dagger \right\rangle = l. \tag{14}$$

Now, passing to the limit in (14) as $n = m \to \infty$ and after that $n = k \to \infty$, we obtain

$$\left\langle x^\dagger, \bar{x} - x^\dagger \right\rangle = \lim_{m\to\infty} \left\langle z_m, \bar{x} - x^\dagger \right\rangle = l = \lim_{k\to\infty} \left\langle z_k, \bar{x} - x^\dagger \right\rangle = \left\langle \bar{x}, \bar{x} - x^\dagger \right\rangle.$$

Hence, $||\bar{x} - x^\dagger||^2 = 0$ or $\bar{x} = x^\dagger$. This says that the whole sequence $\{z_n\}$ converges weakly to $x^\dagger$. Therefore, from Lemma 3.3(ii), we can conclude that the sequence $\{x_n\}$ converges weakly to $x^\dagger$.

Finally, we prove $x^\dagger = \lim_{n\to\infty} P_\Omega(x_n)$. Recalling the relation (9)

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 + (1 + \mu_n)\delta_n, \ \forall x^* \in \Omega \tag{15}$$

and substituting $x^* = P_\Omega(x_n) \in \Omega$ into (15), we obtain

$$||x_{n+1} - P_\Omega(x_n)||^2 \leq ||x_n - P_\Omega(x_n)||^2 + (1 + \mu_n)\delta_n. \tag{16}$$

Since Ω is convex, we get from the definition of the metric projection that

$$||x_{n+1} - P_\Omega(x_{n+1})||^2 \leq ||x_{n+1} - z||^2, \ \forall z \in \Omega,$$

which, with $z = P_\Omega(x_n) \in \Omega$, implies that

$$||x_{n+1} - P_\Omega(x_{n+1})||^2 \le ||x_{n+1} - P_\Omega(x_n)||^2. \tag{17}$$

Combining the relations (16) and (17), we come to the following estimate,

$$||x_{n+1} - P_\Omega(x_{n+1})||^2 \le ||x_n - P_\Omega(x_n)||^2 + (1+\mu_n)\delta_n, \tag{18}$$

or $a_{n+1} \le a_n + (1+\mu_n)\delta_n$ where $a_n = ||x_n - P_\Omega(x_n)||^2$. Since $\sum_{n=1}^{\infty}(1+\mu_n)\delta_n < +\infty$, from Lemma 2.3, we see that the sequence $\{a_n\}$ converges as $n \to \infty$.

For each $n \ge 1$, let $b_n = P_\Omega(x_n)$. Then, the sequence $\{b_n\}$ converges to some $b \in H_1$. Indeed, for each $n \ge 1$ and $p \ge 1$, it follows from Lemma 2.1(ii), the definition of b_n, and the relation (15) that

$$\begin{aligned}
||b_{n+p} - b_n||^2 &= ||P_\Omega(x_{n+p}) - P_\Omega(x_n)||^2 \\
&\le ||x_{n+p} - P_\Omega(x_n)||^2 - ||x_{n+p} - P_\Omega(x_{n+p})||^2 \\
&\le \big(||x_{n+p-1} - P_\Omega(x_n)||^2 + (1+\mu_{n+p-1})\delta_{n+p-1}\big) - a_{n+p} \\
&\le ||x_{n+p-2} - P_\Omega(x_n)||^2 + (1+\mu_{n+p-2})\delta_{n+p-2} \\
&\quad +(1+\mu_{n+p-1})\delta_{n+p-1} - a_{n+p} \\
&\le \dots \\
&\le ||x_n - P_\Omega(x_n)||^2 + \sum_{t=n}^{n+p-1}(1+\mu_t)\delta_t - a_{n+p} \\
&= (a_n - a_{n+p}) + \sum_{t=n}^{n+p-1}(1+\mu_t)\delta_t.
\end{aligned}$$

Passing to the limit in the last inequality as $n,\ p \to \infty$ and noting that $\sum_{n=1}^{\infty}(1+\mu_n)\delta_n < +\infty$, we obtain

$$\lim_{n,\ p\to\infty} ||b_{n+p} - b_n||^2 = 0.$$

Thus, the sequence $\{b_n\}$ is a Cauchy sequence in H_1, i.e., there exists $b \in H_1$ such that $\lim_{n\to\infty} b_n = b$. From $b_n = P_\Omega(x_n)$ and Lemma 2.1(iii), we obtain

$$\langle x^\dagger - b_n, x_n - b_n \rangle \le 0. \tag{19}$$

Passing to the limit in (19) as $n \to \infty$, we find that $||x^\dagger - b||^2 = \langle x^\dagger - b, x^\dagger - b\rangle \le 0$. Thus $b = x^\dagger$ or $x^\dagger = \lim_{n\to\infty} P_\Omega(x_n)$. This finishes the proof. □

4. **Further remarks.** In this section, we present several remarks regarding the assumptions of Theorem 3.4 in the previous section and an extension of Algorithm 1 in the case when f and F can be splitted into several bifunctions. We begin with assumption A3.

Remark 3. Assumption A3 has been also considered by the authors in [27, 36, 40]. This assumption is used to prove that the subgradient sequence $\{g_n\}$ is bounded when $\{z_n\}$ is bounded (similarly, with the sequence $\{w_n\}$ for bifunction F). We can assume directly as in [36] that the sequences $\{g_n\}$ and $\{w_n\}$ are bounded. However, from the proofs of Lemma 3.3(iii) and Theorem 3.4, we see that, without assumption A3, the result in this paper is still true if f and F are jointly weakly continuous on two open sets containing C and Q, respectively, see, e.g. [38, Proposition 4.3].

In the next remark, by an example, we show that assumption A4 is necessary in the formulation of Theorem 3.4.

Remark 4. Algorithm 1 converges under the assumption that f, F satisfy paramonotone condition A4. The following simple example implies that, without this condition, the iterative sequence generated by the algorithm cannot converge (weakly) to any solution of the problem. Indeed, consider our problem with $C = Q = H_1 = H_2 = \Re^2$, $A = I$ and $f(x, y) = F(x, y) = x_1y_2 - x_2y_1$ for all $x,\ y \in \Re^2$. The problem has an unique solution $x^* = (0, 0)^T$. Assumptions A1-A3 are automatically satisfied for f and F. However, the hypothesis A4 does not hold. Indeed, we have that $f(y, x^*) = F(y, Ax^*) = 0$ for all $y \in C = Q = \Re^2$ which cannot imply that $y \in EP(f, C)$ or $y \in EP(F, Q)$. Now, by some computation, from Algorithm 1, we obtain for each $n \geq 0$ and $x_n = (x_{1n}, x_{2n})^T \in \Re^2$ that

$$y_n = (x_{1n} + \gamma_n x_{2n}, x_{2n} - \gamma_n x_{1n})^T,$$
$$z_n = (x_{1n} + \mu_n\gamma_n x_{2n}, x_{2n} - \mu_n\gamma_n x_{1n})^T.$$

Thus, from the definiton of x_{n+1}, we obtain for each $n \geq 0$,

$$x_{n+1} = ((1-\mu_n\alpha_n\gamma_n)x_{1n} + (\mu_n\gamma_n+\alpha_n)x_{2n}, (-\mu_n\gamma_n-\alpha_n)x_{1n} + (1-\mu_n\alpha_n\gamma_n)x_{2n})^T.$$

By setting $a_n = 1 - \mu_n\alpha_n\gamma_n$ and $b_n = \mu_n\gamma_n + \alpha_n$, x_{n+1} can be shortly rewritten as follows:

$$x_{n+1} = (a_n x_{1n} + b_n x_{2n}, -b_n x_{1n} + a_n x_{2n})^T.$$

This implies that

$$||x_{n+1}||^2 \;=\; (a_n x_{1n} + b_n x_{2n})^2 + (-b_n x_{1n} + a_n x_{2n})^2 = (a_n^2 + b_n^2)||x_n||^2.$$

On the other hand, it follows from the definitions of a_n and b_n that

$$a_n^2 + b_n^2 = 1 + \mu_n^2\alpha_n^2\gamma_n^2 + \mu_n^2\alpha_n^2 + \gamma_n^2 > 1.$$

Therefore $||x_{n+1}||^2 > ||x_n||^2$ for each $n \geq 0$, which implies, by the induction, that $||x_{n+1}||^2 > ||x_0||^2$. Thus, $\lim_{n\to\infty} ||x_{n+1}||^2 > 0$, provided that $x_0 \neq 0$. This says that the sequence $\{x_n\}$ cannot converge to the solution $x^* = (0, 0)^T$ of the problem. Since the weak convergence and strong convergence are the same in finite dimensional spaces, the sequence $\{x_n\}$ cannot converge weakly to the solution $x^* = (0, 0)^T$.

Remark 5. The convergence of Algorithm 1 can be ensured under the assumption that the solution set Ω of problem (SEP) is nonempty. We remark here that, without this assumption, the algorithm can diverge. It is sufficient to consider our problem with $H_1 = H_2 = \Re^2$, $C = \{(x, 0) \in H_1 : x \geq 1\}$, $Q = \left\{(x, y) \in H_2 : x \geq 1,\ y \geq \frac{1}{\sqrt{x}}\right\}$, the operator $A = I$, and the two bifunctions $f(x, y) = \delta_C(y) - \delta_C(x)$ for all $x, y \in C$, and $F(x, y) = \delta_Q(y) - \delta_Q(x)$ for all $x,\ y \in Q$, where δ_C and δ_Q are the indicator functions to C and Q, respectively. It is easy to see that the solution set of problem (SEP) is $\Omega = C \cap Q = \emptyset$. Note that the projection of any point in C onto Q is always on the boundary of Q. Assume that at iteration n, we have $x_n = (x_{1n}, 0)^T \in C$ with $x_{1n} \geq 1$. From Algorithm 1 and $A = I$, we see that $u_n = P_Q(Ax_n) = P_Q(x_n)$. Since u_n is on the boundary of Q, it is of the form $u_n = \left(u_{1n}, \frac{1}{\sqrt{u_{1n}}}\right) \in Q$, where u_{1n} is the unique solution of the strongly convex problem $\min_{t\geq 1} ||a - x_n||^2$ with $a = (t, \frac{1}{\sqrt{t}})$, or

$$\min_{t\geq 1} \left\{ h(t) = (t - x_{1n})^2 + \frac{1}{t} \right\}. \tag{20}$$

We have that $h'(t) = 2(t - x_{1n}) - \frac{1}{t^2}$. By a straightforward computation, we see that $h'(x_{1n} + \frac{1}{4x_{1n}^2}) < 0$. Thus, the unique optimal solution u_{1n} of problem (20) must satisfy the inequality $u_{1n} > x_{1n} + \frac{1}{4x_{1n}^2}$. Since $0 \in \partial f_2(x,x)$ and $0 \in \partial F_2(x,x)$ for all x, we can choose $w_n = g_n = 0 \in \Re^2$. Moreover, we can take $\mu_n = \frac{1}{2} \in (0,1) = (0, \frac{1}{||A||^2})$. Thus, from Algorithm 1, we obtain that $y_n = P_Q(u_n) = u_n$, $z_n = P_C\left(\frac{x_n+y_n}{2}\right)$ and $x_{n+1} = P_C(z_n) = P_C\left(\frac{x_n+y_n}{2}\right) = P_C\left(\frac{x_n+u_n}{2}\right) = (\frac{u_{1n}+x_{1n}}{2}, 0)^T$. This together with the inequality $u_{1n} > x_{1n} + \frac{1}{4x_{1n}^2}$ implies that

$$x_{1,n+1} = \frac{u_{1n} + x_{1n}}{2} > x_{1n} + \frac{1}{8x_{1n}^2}, \ \forall n \geq 0.$$

Thus, it is not difficult to see that $x_{1n} \to +\infty$. Hence $||x_n|| = |x_{1n}| \to +\infty$ as $n \to \infty$. This says that the sequence $\{x_n\}$ generated by Algorithm 1 diverges.

Remark 6. Algorithm 1 can be extended to the case when $f = \sum_{i=1}^N f_i$ and $F = \sum_{j=1}^M F_j$. In that case, the parallel projection algorithm is given by

$$\begin{cases} u_n = P_Q(Ax_n), \ w_n^j \in \partial_{\epsilon_n} F_j(u_n,.)(u_n), \\ \gamma_n = \frac{\beta_n}{\max\{\rho_n, ||w_n^1||, \ldots, ||w_n^M||\}}, \ y_n^j = P_Q(u_n - \gamma_n w_n^j), \\ y_n = \frac{1}{M}\sum_{j=1}^M y_n^j, \ z_n = P_C\left(x_n + \mu_n A^*(y_n - Ax_n)\right), \\ g_n^i \in \partial_{\epsilon_n} f_i(z_n,.)(z_n), \ \alpha_n = \frac{\beta_n}{\max\{\rho_n, ||g_n^1||, \ldots, ||g_n^N||\}}, \\ x_n^i = P_C(z_n - \alpha_n g_n^i), x_{n+1} = \frac{1}{N}\sum_{i=1}^N x_n^i. \end{cases} \tag{21}$$

Under the assumptions as in Theorem 1, the sequence $\{x_n\}$ generated by (21) converges weakly to some solution $x^\dagger$ of problem (SEP). Moreover, $x^\dagger = \lim_{n\to\infty} P_\Omega(x_n)$.

Remark 7. Algorithm 1 is performed under the previous knowledge of the norm of operator A. An open question, is then to design an algorithm which can be used without the prior knowledge of the operator norm as, for instance, in [21, Algorithm 4.1] for problem (SEP) or in [31] for the split feasibility problem.

5. **Split common equilibrium problems.** This section deals with an extension of Algorithm 1 for solving the split common equilibrium problem (SCEP) considered in [7, 18, 20]. This problem is stated as follows:

$$\begin{cases} \text{Find } x^* \in C \text{ such that } f_i(x^*, y) \geq 0, \ \forall y \in C, \ i \in I, \\ \text{and } u^* = Ax^* \in Q \text{ solves } F_j(u^*, u) \geq 0, \ \forall u \in Q, \ j \in J, \end{cases} \tag{SCEP}$$

where $I = \{1, 2, \ldots, N\}$, $J = \{1, 2, \ldots, M\}$; C and Q are two nonempty closed convex subsets of two real Hilbert spaces H_1, H_2, respectively; $A : H_1 \to H_2$ is a bounded linear operator; and $f_i : C \times C \to \Re$ and $F_j : Q \times Q \to \Re$ are bifunctions with $f_i(x,x) = 0$ for all $x \in C$ and $F_j(u,u) = 0$ for all $u \in Q$. We denote here by Ω the solution set of problem (SCEP) and assume that it is nonempty. It is well known that problem (SCEP) contains properly many split-like problems, see, e.g., [7]. For solving problem (SCEP), He [18] used the resolvent of a bifunction (the proximal point method) to propose a weakly convergent parallel algorithm [18, algorithm (3.2)] in the case $N > 1$ and $M = 1$. In a different direction, the author in [20] has additionally incorporated in the previous algorithm the extended extragradient method and has proposed two weakly and strongly convergent parallel algorithms. In this section, as an extension of Algorithm 1, we present a different algorithm, which only uses the projections to design.

In order to solve problem (SCEP), we also assume that for each $i \in I$ and $j \in J$, the bifunctions f_i and F_j have the same properties as f and F in Section 3. The algorithm is designed as follows:

Algorithm 2 (Parallel algorithm for SCEPs)**.** .

Initialization: Choose $x_0 \in C$ and the parameter sequences $\{\rho_n\}$, $\{\beta_n\}$, $\{\epsilon_n\}$, $\{\mu_n\}$ such that condition C1-C3 above hold. Moreover, consider additionally the sequences $\left\{\theta_n^j\right\}$, $\left\{\tau_n^i\right\} \subset [b,c] \subset (0,1)$ such that $\sum_{j\in J} \theta_n^j = \sum_{i\in I} \tau_n^i = 1$.

Iterative Steps: Assume that $x_n \in C$ is known, calculate x_{n+1} as follows:

Step 1. Select $w_n^j \in \partial_{\epsilon_n} F_j(u_n,.)(u_n)$ where $u_n = P_Q(Ax_n)$, and compute

$$\gamma_n^j = \frac{\beta_n}{\max\left\{\rho_n, ||w_n^j||\right\}}, \qquad y_n^j = P_Q(u_n - \gamma_n^j w_n^j), \qquad y_n = \sum_{j\in J} \theta_n^j y_n^j.$$

Step 2. Compute $z_n = P_C\left(x_n + \mu_n A^*(y_n - Ax_n)\right)$.

Step 3. Select $g_n^i \in \partial_{\epsilon_n} f_i(z_n,.)(z_n)$ and compute

$$\alpha_n^i = \frac{\beta_n}{\max\{\rho_n, ||g_n^i||\}}, \qquad x_n^i = P_C(z_n - \alpha_n^i g_n^i), \qquad x_{n+1} = \sum_{i\in I} \tau_n^i x_n^i.$$

Set $n := n+1$ and go back to **Step 1**.

We omit here the proof of convergence of Algorithm 2. In fact, it is easy to obtain it by repeating the proofs in the previous section. We have the following result.

Theorem 5.1. *The sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to some solution $x^\dagger$ of problem (SCEP). Moreover, $x^\dagger = \lim_{n\to\infty} P_\Omega(x_n)$.*

6. **Computational experiments.** This section presents several experiments to illustrate the numerical behavior of Algorithm 1 (shortly, PM) and also to compare it with the behaviors of other well known algorithms. The test problem here can be considered as an extension of the Nash-Cournot oligopolistic equilibrium model in [11, 15] to the split equilibrium model in [21]. More precisely, the problem is for $H_1 = \Re^m$ and $H_2 = \Re^k$. The bifunction f on H_1 is of the form

$$f(x,y) = \left\langle \bar{M}x + \bar{N}y + p, y - x\right\rangle,$$

where p is a vector in $\Re^m$ and $\bar{M}$, $\bar{N}$ are two matrices of order m such that $\bar{N}$ is symmetric positive semidefinite and $\bar{N} - \bar{M}$ is negative semidefinite. The bounded linear operator $A : \Re^m \to \Re^k$ is defined by a matrix of size $k \times m$. All the entries of A are generated randomly (and uniformly) in $[-10, 10]$. The bifunction F also has the following form

$$F(x,y) = \langle Mx + Ny + q, y - x\rangle$$

where q is a vector in $\Re^k$ and M, N are two matrices of order k such that N is symmetric positive semidefinite and $N - M$ is negative semidefinite. Two feasible sets respectively are $C = [-1,5]^m$ and $Q = [-2,5]^k$. In the purpose that the solution set of the problem is nonempty and that all the algorithms can work, the two vectors p and q are chosen as the two zero vectors in $\Re^m$ and $\Re^k$, respectively. The matrices $\bar{M}$ and $\bar{N}$ are generated randomly to satisfy the conditions[1] (the matrices $M,\ N$ are also generated randomly at this way).

This section is divided into two parts: Subsection 6.1 studies the numerical behavior of Algorithm 1, while Subsection 6.2 reports several results in comparing Algorithm 1 with other algorithms, namely the Extragradient-Proximal Method (EGPM) in [20, Algorithm 1]; the Hybrid Extragradient- Proximal Method (HEGPM) in [20, Algorithm 2]; the Projected Subgradient-Proximal Method (PSPM) in [21, Algorithm 3.1]; and the Modified Projected Subgradient-Proximal Method (MPSPM) in [21, Algorithm 4.1]. The solution of the considered problem is $x^* = 0$ and it is easy to see that conditions A1-A4 are satisfied. Thus, also as in [21], all the algorithms can be applied. We have used the sequence $D_n = ||x_n - x^*||^2$, $n = 0, 1, \ldots$ to study the convergence of all the algorithms. The starting point is $x_0 = (1, 1, \ldots, 1)^T \in H_1$. The convergence of D_n to 0 implies that the sequence $\{x_n\}$ generated by each algorithm converges to the solution x^* of the problem.

All the projections and the optimization problems are solved effectively by using the function *quadprog* in the Matlab 7.0 Optimization Toolbox. All the programs are written in Matlab and computed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50 GHz, RAM 2.00 GB.

6.1. **Numerical behavior of Algorithm 1.** In this part, the four matrices M, N, $\bar{M}$ and $\bar{N}$ are generated ramdomly. In this case, it is not easy to implement the four algorithms EGPM, HEGPM, PSPM, MPSPM because they use the resolvent mapping which in general is difficult to compute. Then, we only illustrate the numerical behavior of Algorithm 1. The six sequences of $\{\beta_n\}$ are taken as $\beta_n = \frac{1}{(n+1)^s}$, $s \in \{1;\ 0.9;\ 0.8;\ 0.7;\ 0.6;\ 0.51\}$. Other parameters are $\epsilon_n = 0$, $\rho_n = 1$, $\mu_n = \frac{1}{||A||^2}$. Figures 1 - 4 show the behavior of $\{D_n\}$ generated by Algorithm 1 for different pairs of (m, k). In each figure, the y-axis represents the value of D_n while the x-axis is for the execution time elapsed in second. In view of these figures, we see that the rate of convergence of Algorithm 1 depends strictly on the rate of convergence of the sequence $\{\beta_n\}$. Algorithm 1 in general works well for the sequences $\beta_n = \frac{1}{(n+1)^s}$ with $s \in \{0.8,\ 0.7,\ 0.6,\ 0.51\}$, and it is especially noted that in all the cases the new algorithm works badly for the natural sequence $\beta_n = \frac{1}{n+1}$.

6.2. **Comparison of Algorithm 1 with other algorithms.** Four aforementioned algorithms EGPM, HEGPM, PSPM and MPSPM have been designed from the resolvent T_r^F of the bifunction F. In order to compute easily the value of the resolvent mapping T_r^F, we have chosen $M = N$. In that case, the resolvent T_r^F of F coincides with the proximal mapping of the function $g(x) = \langle Mx, x\rangle$ for $r > 0$, i.e., $T_r^F(u) = \text{prox}_{rg}(u)$, where

$$\text{prox}_{rg}(u) = \arg\min\left\{g(v) + \frac{1}{r}||v - u||^2 : v \in Q\right\}.$$

[1]Choose randomly $\lambda_{1k} \in [-10, 0]$, $\lambda_{2k} \in [1, 10]$ for all $k = 1, \ldots, m$. Set $\widehat{Q}_1$, $\widehat{Q}_2$ as two diagonal matrixes with eigenvalues $\{\lambda_{1k}\}_{k=1}^m$ and $\{\lambda_{2k}\}_{k=1}^m$, respectively. Then, we consider a positive semidefinite matrix $\bar{N}$ and a negative semidefinite matrix T by using full random orthogonal matrixes with $\widehat{Q}_2$ and $\widehat{Q}_1$, respectively. Finally, set $\bar{M} = \bar{N} - T$

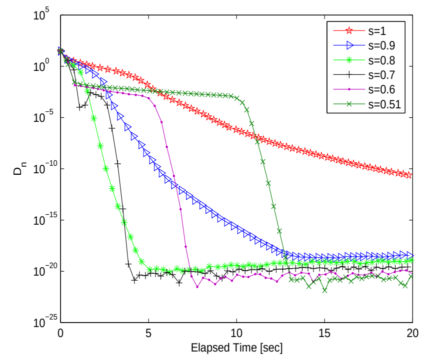

FIGURE 1. Algorithm 1 for $(m, k) = (30, 20)$ and different sequences of β_n. The number of iterations is 360, 353, 339, 360, 355, 376, respectively.

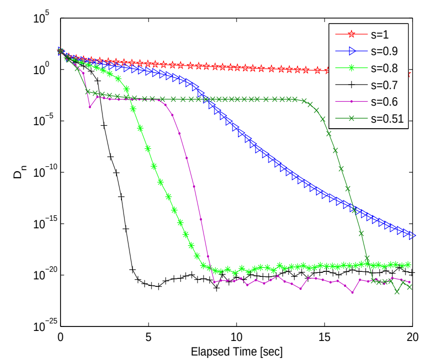

FIGURE 2. Algorithm 1 for $(m, k) = (60, 40)$ and different sequences of β_n. The number of iterations is 258, 333, 336, 326, 291, 293, respectively.

The mapping $\text{prox}_{rg}(u)$ can be effectively computed by using the Optimization Toolbox in Matlab. Moreover, it is emphasized that although the conditions of convergence of the four compared algorithms in general are different to the ones of Algorithm 1, we still wish to present a numerical comparison between them. For implementing algorithms EGPM and HEGPM [20], we need two Lipschitz-type constants c_1 and c_2 of f (they are $c_1 = c_2 = ||\bar{M} - \bar{N}||/2$). The parameters have been chosen in all the experiments as follows:

(i) $\lambda = \frac{1}{5c_1}$, $\mu_n = \mu = \frac{1}{||A||^2}$ for EGPM, HEGPM, PSPM and Algorithm 1 (PM).

(ii) $\epsilon_n = 0$, $\rho_n = 1$, $\beta_n = \frac{1}{(n+1)^{0.7}}$ for PSPM, MPSPM and Algorithms 1.

(iii) $\nu_n = 3$ for MPSPM and $r_n = 1$ for EGPM, HEGPM, PSPM, MPSPM.

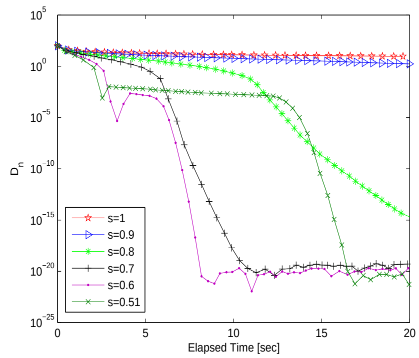

FIGURE 3. Algorithm 1 for $(m, k) = (100, 50)$ and different sequences of β_n. The number of iterations is 215, 236, 283, 280, 321, 290, respectively.

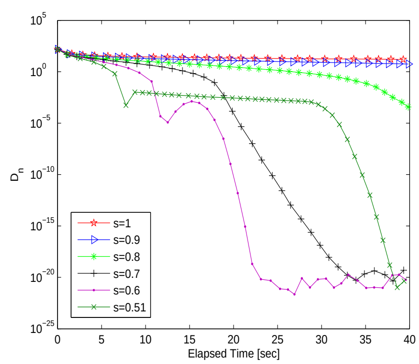

FIGURE 4. Algorithm 1 for $(m, k) = (150, 100)$ and different sequences of β_n. The number of iterations is 161, 188, 219, 209, 245, 264, respectively.

Figures 5 - 8 describe the behavior of the sequence $\{D_n\}$ generated by the algorithms. In view of this, we see that the proposed algorithm has competitive advantages over existing algorithms. It is also seen that the obtained error from Algorithm 1 is better than the one from other algorithms.

7. **Conclusions.** The paper has considered a class of split inverse problems involving equilibrium problems in Hilbert spaces, so-called briefly the split equilibrium problem. This problem unifies in a simple form various previously known split-type problems. A new algorithm, which only uses the projections to design, has been proposed for approximating the solutions. A theorem of weak convergence has been proved under suitable conditions. The convergent conditions are also discussed, and as be seen, they are almost necessary in the formulation of the convergence theorem. Several extensions of the resulting algorithm to the split common equilibrium problem have been also presented in the paper. The numerical behavior of the new algorithm is studied by reporting some numerical experiments. In particular, it

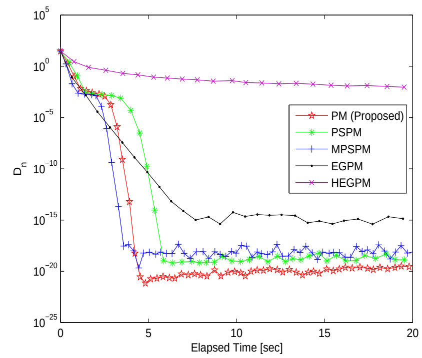

FIGURE 5. Experiment for the algorithms with $(m, k) = (30, 20)$. The number of iterations is 334, 240, 379, 168, 130, respectively.

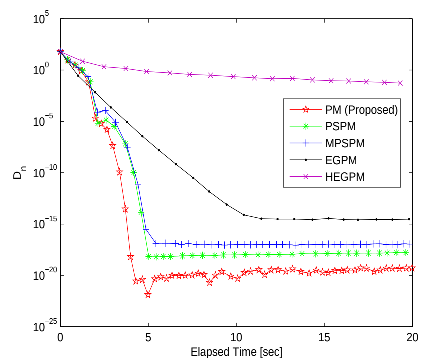

FIGURE 6. Experiment for the algorithms with $(m, k) = (60, 40)$. The number of iterations is 326, 221, 292, 129, 108, respectively.

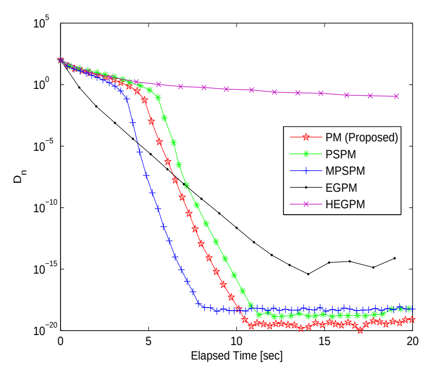

FIGURE 7. Experiment for the algorithms with $(m, k) = (100, 50)$. The number of iterations is 308, 250, 356, 114, 89, respectively.

is seen that the proposed algorithm also has competitive advantages over existing methods.

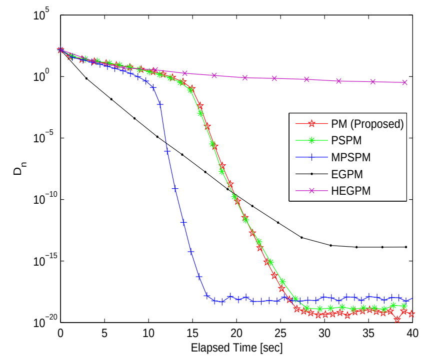

FIGURE 8. Experiment for the algorithms with $(m, k) = (150, 100)$. The number of iterations is 254, 192, 271, 87, 69, respectively.

Acknowledgments. The author would like to thank the Associate Editor and the two anonymous referees for their valuable comments and suggestions which helped us very much in improving the original version of this paper. This work is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under the project: 101.01-2017.315

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