Fixed Point Theorems with Applications

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As a very important part of nonlinear analysis, fixed point theory plays a key role in the solvability of many complex systems from mathematics applied to chemical reactors, neutron transport, population biology, infectious diseases, economics, applied mechanics, and more.

The main aim of *Fixed Point Theorems with Applications* is to explain new techniques for investigation of different classes of ordinary and partial differential equations. The development of the fixed point theory parallels the advances in topology and functional analysis. Recent research investigated not only the existence but also the positivity of solutions for various types of nonlinear equations. This book will be of interest to those working in functional analysis and its applications.

Combined with other nonlinear methods such as variational methods and the approximation methods, the fixed point theory is powerful in dealing with many nonlinear problems from the real world.

The book can be used as a textbook to develop an elective course on nonlinear functional analysis with applications in undergraduate and graduate programs in mathematics or engineering programs.

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Contents

Preface

As a very important part of nonlinear analysis, fixed point theory plays a key role with regards to the solvability of many complex systems from applied mathematics (chemicals reactors, neutron transport, population biology, infection diseases, economics, applied mechanics, and more). The development of the fixed point theory parallels the advances in topology and functional analysis. During the last couple of years, the theory developed quickly in many directions starting from Brouwer's fixed point theorem (1910), Banach's contraction principle (1922), and Schauder's fixed point theorem for compact mappings (1930). An extension and a combination of the last two results to the sum of a contraction and a compact mapping were discovered by Krasnosel'skii in 1955. Indeed, many problems in science can be mathematically recast as nonlinear equations of the form $Tx + Fx = x$ and posed in some closed convex subset of a Banach space. Notice further that the positivity of solutions of nonlinear equations, especially ordinary, fractional, partial differential equations, and integral equations, is a very important issue in applications where a positive solution may represent a density, temperature, velocity, density, gravity, and more. That is why many recent research investigated not only the existence but also the positivity of solutions for various types of nonlinear equations. Combined with other nonlinear methods such that the variational methods, the approximation methods, and more, the fixed point theory turns out to be powerful in dealing with many nonlinear problems from the real world.

This book is devoted to the study of the existence, multiplicity, positivity, and localization of fixed points for some operators that are of the form $T + F$ where $(I - T)$ is a Lipschitz invertible mapping and F is a k-set contraction. The text offers the reader an overview of recent developments of positive fixed point theorems and their applications.

The book consists of six chapters. In Chapter 1, some preliminaries and basic concepts used throughout this book are collected. The chapter opens with the topology of the normed and Banach spaces and the linear operators defined therein. However, the measure of noncompactness (MNC for short) of a set in a metric space occupies the major part of this chapter. The abstract definition is followed by the classical Kuratowski and Hausdorff measures where the fundamental properties are investigated in detail. MNCs are involved in many convergence results in ordinary differential equations (ODEs) and partial differential equations (PDEs), particularly to derive compactness of some nonlinear operators. MNCs are also connected with strict-set contractions, 1-set contraction and condensing maps for which numerous fixed point theorems have been established so far. Chapter 1 ends with some useful compactness criteria that can be regarded as extensions of the classical Ascoli-Arzelà Theorem.

The fixed point index is a generalization of the Leray-Schauder degree. Chapter 2 starts with a reminder of the main properties of the fixed point index for strict set contractions set on bounded convex or on translates of cones of some Banach spaces. As a consequence, the celebrated Krasnosel'skii's compression–expansion fixed point theorem is derived. However, the core of Chapter 2 is the fixed point index for some classes of sums of two mappings, one being h-expansive and the other k-set contraction. The definition of a generalized fixed point index as well as some of its properties are presented. Then several conditions allowing computation of this index are shown.

In Chapter 3, there are formulated and proved various theorems of existence of positive fixed points for the sum of two operators. In particular, several new versions of Krasnosel'skii's expansion-compression fixed point theorems type are presented (fixed point in conical annulus, extensions to more general region of cones and more general class of mappings, extensions to translates of cones, and more) as well as a vector version useful in regard to the solvability of some nonlinear systems. The chapter ends with the proof of a new version of the original Leggett-Williams fixed point theorem and some fixed point theorems on open sets of cones for special mappings that include 1-set contractions.

The last three chapters are exclusively concerned with some applications of the theory developed in Chapters 2–3 to some ordinary and partial differential equations.

Mathematical problems formulated as ODEs are investigated in Chapter 4. We consider the existence of the classical solutions for some classes of systems ODEs (and even nth-order ODEs) and some boundary value problems for ODEs. The questions of periodicity, positivity, and multiplicity of solutions are investigated.

Chapter 5 mainly discusses some classes of parabolic equations. Some criteria for the existence of classical solutions for some classes' initial boundary value problems for nonlinear parabolic equations are presented. Subsequently, an application for existence of classical solutions for the Burgers-Fisher equation is provided. Existence of positive solutions is also proven.

As for the final Chapter 6, it deals with applications with some applications of the index fixed point theory to the solvability of some classes of initial value problems (IVPs) associated with some hyperbolic equations represented by the wave equation in different dimensions.

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1

Preliminaries

1.1 Normed Linear Spaces

Definition 1.1.1 *A set X with a collection $\mathscr{F}$ of its subsets is called a topological space if $\mathscr{F}$ possesses the following properties.*

1. $\emptyset, X \in \mathscr{F}$.
2. *An intersection of a finite number of sets of $\mathscr{F}$ belongs to $\mathscr{F}$.*
3. *A union of any subcollection of $\mathscr{F}$ belongs to $\mathscr{F}$.*

The elements of $\mathscr{F}$ are called open sets. A subset $U \subset X$ is called a neighborhood of a point $x \in X$ if there is an open set $G \subset X$ such that $x \in G \subset U$. The collection $\mathscr{F}$ is called a topology of X.

Example 1.1.2 *Let X be a given set. Take $\mathscr{F} = \{\emptyset, X\}$. Then X is a topological space. The topology $\mathscr{F}$ is the trivial topology.*

Example 1.1.3 *The discrete topology on X is defined by letting every subset of X be open.*

Definition 1.1.4 *A set X with a real-valued function $d : X \times X \to [0, \infty)$ for which*

1. *(identity of indiscernibles)* $d(x,y) = 0$ *if and only if* $x = y \in X$,
2. *(symmetry property)* $d(x,y) = d(y,x)$ *for any* $x, y \in X$,
3. *(triangle inequality)* $d(x,y) \leq d(x,z) + d(z,y)$ *for any* $x, y, z \in X$,

is called a metric space. The function d is called a metric of X.

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Example 1.1.5 *The set of real numbers is a metric space with a metric*

$$d(x,y) = |x-y|, \quad x,y \in \mathbb{R}.$$

Example 1.1.6 *The set of positive real numbers with a metric*

$$d(x,y) = \left|\log \frac{y}{x}\right|, \quad x,y \in (0,\infty),$$

is a metric space.

Example 1.1.7 *The set $\mathscr{C}([a,b])$ of continuous functions on $[a,b]$ with a metric*

$$d(f,g) = \max_{x\in[a,b]} |f(x)-g(x)|, \quad f,g \in \mathscr{C}([a,b]).$$

Definition 1.1.8 *If d is a metric on X, then*

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

is called an open ball centered at x with radius $r > 0$.

Definition 1.1.9 *Open sets in a metric space X are defined as subsets $G \subset X$ which have the following property: for every $x \in X$, there is a $\delta > 0$ such that $B(x,\delta) \subset G$.*

Definition 1.1.10 *A subset G of a topological space X is called a closed set if $X\backslash G$ is open.*

Definition 1.1.11 *If $A \subset X$, then the intersection of all closed sets containing A is called the closure of A and it is denoted by $\overline{A}$, i.e.,*

$$\overline{A} = \bigcap\{B : \quad A \subset B, \quad B \quad is \quad closed\}.$$

A dual notion is the interior of A, i.e.,

$$\overset{\circ}{A} = \bigcup\{B : \quad B \subset A, \quad B \quad is \quad open\}.$$

The boundary ∂A of A is defined by

$$\partial A = \overline{A} \bigcap \overline{(X\backslash A)}.$$

Definition 1.1.12 *A subset A of X is said to be dense if $\overline{A} = X$.*

In a metric space X, we have the following equivalences.

1. $x \in \overline{A}$ if and only if there exists $\{x_n\}_{n=1}^{\infty} \subset A$ so that $\lim_{n\to\infty} d(x_n, x) = 0$.
2. $x \in \overset{\circ}{A}$ if and only if there exists a $\delta > 0$ such that $B(x, \delta) \subset A$.

Definition 1.1.13 *A metric space X is said to be separable if there is a countable dense subset of X.*

Now, we are ready with the main object in this section. It is well known that the notion of norm is of fundamental importance in discussing linear topological spaces. We shall begin with the definition of the semi-norm.

Definition 1.1.14 *A real-valued function ν defined on a vector space E is called a semi-norm on E, if the following conditions are satisfied:*

1. *$\nu(x+y) \leq \nu(x) + \nu(y)$ for any $x, y \in E$ (sub-additivity),*
2. *$\nu(\alpha x) = |\alpha|\nu(x)$ for any $x \in E$ and $\alpha \in \mathbb{R}$ (homogeneity),*
3. *$\nu(x) \geq 0$ for any $x \in E$ (non-negativity).*

Theorem 1.1.15 *If E is a real vector space and $\nu : E \mapsto \mathbb{R}$ is a semi-norm. Then*

$$\nu(x-y) \geq |\nu(x) - \nu(y)|,$$

for any $x, y \in E$.

Proof 1.1.16 *We have*

$$\nu(x) \leq \nu(x-y) + \nu(y)$$

for any $x, y \in E$, so

$$\nu(x) - \nu(y) \leq \nu(x-y) \tag{1.1}$$

for any $x,y \in E$. Since

$$\begin{aligned} \nu(x-y) &= |-1|\nu(y-x) \\ &\geq \nu(y)-\nu(x) \end{aligned}$$

for any $x,y \in E$, we have

$$-(\nu(x)-\nu(y)) \leq \nu(x)-\nu(y) \tag{1.2}$$

for any $x,y \in E$. Inequalities (1.1) and (1.2) give the desired inequality.

Definition 1.1.17 *A normed space is an ordered pair $(E, \|\cdot\|)$, where E is a vector space (also called a linear space) over F and $\|\cdot\|$ is a norm on E, i.e., a function $\|\cdot\| : E \longmapsto \mathbb{R}$ such that for any $x,y,z \in E$ the following holds.*

1. *$\|x\| \geq 0$ (non-negativity),*
2. *$\|x\| = 0$ iff $x = 0$ (separate points),*
3. *$\|\lambda x\| = |\lambda|\|x\|$ for any $\lambda \in F$ (homogeneity of the norm),*
4. *$\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality).*

Note that the first condition follows from the other three. To see this, take $x \in E$ arbitrarily. Then

$$\begin{aligned} \|0\| &= \|x+(-x)\| \leq \|x\| + \|-x\| = \|x\| + \|(-1)x\| \\ &= \|x\| + |-1|\|x\| = 2\|x\|. \end{aligned}$$

Hence, using the second condition, we get that $\|x\| \geq 0$.

Example 1.1.18 *In E_n (n-dimensional Euclidean space) define a norm as follows.*

$$||x|| = \left(\sum_{l=1}^{n} |x_l|^2\right)^{\frac{1}{2}}, \quad x_l \in F, \quad l \in \{1,\ldots,n\}, \quad x = (x_1,\ldots,x_n). \tag{1.3}$$

We will check that (1.3) satisfies all axioms for a norm. Take $x,y \in E_n$, $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$, arbitrarily. Then

1. $\|x\| \geq 0$.

2. $\|x\| = 0$ *iff* $\left(\sum_{l=1}^{n} |x_l|^2\right)^{\frac{1}{2}} = 0$ *iff* $x_l = 0$ *for any* $l \in \{1, \ldots, n\}$.

3.

$$\begin{aligned} \|\lambda x\| &= \left(\sum_{l=1}^{n} |\lambda x_l|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{l=1}^{n} |\lambda|^2 |x_l|^2\right)^{\frac{1}{2}} \\ &= |\lambda| \left(\sum_{l=1}^{n} |x_l|^2\right)^{\frac{1}{2}} \\ &= |\lambda| \|x\|, \end{aligned}$$

for any $\lambda \in F$.

4. Applying Minkowski's inequality, we get

$$\begin{aligned} \|x+y\| &= \left(\sum_{l=1}^{n} |x_l + y_l|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{l=1}^{n} |x_l|^2\right)^{\frac{1}{2}} + \left(\sum_{l=1}^{n} |y_l|^2\right)^{\frac{1}{2}} \\ &= \|x\| + \|y\|. \end{aligned}$$

Example 1.1.19 *In the space* $\mathscr{C}^k([a,b])$ *define a norm*

$$\|f\| = \sum_{l=0}^{k} \max_{t \in [a,b]} \left| f^{(l)}(t) \right|, \quad f \in \mathscr{C}^k([a,b]). \tag{1.4}$$

We will check that (1.4) satisfies all axioms for a norm. Let $f, g \in \mathscr{C}^k([a,b])$ *and* $\lambda \in F$ *be chosen arbitrarily. Then*

1. $\|f\| \geq 0.$

2. $0 = \|f\| \quad iff \quad \sum_{l=0}^{k} \max_{t\in[a,b]} \left| f^{(l)}(t) \right| = 0 \quad iff$

$\max_{t\in[a,b]} \left| f^{(l)}(t) \right| = 0$ *for any* $l \in \{0,\dots,k\}$ *iff* $f \equiv 0$ *on* $[a,b]$.

3.

$$\begin{aligned} ||\lambda f|| &= \sum_{l=0}^{k} \max_{t\in[a,b]} \left| (\lambda f)^{(l)}(t) \right| \\ &= \sum_{l=0}^{k} \max_{t\in[a,b]} \left| \lambda f^{(l)}(t) \right| \\ &= |\lambda| \sum_{l=0}^{k} \max_{t\in[a,b]} \left| f^{(l)}(t) \right| \\ &= |\lambda| \|f\|. \end{aligned}$$

4.

$$\begin{aligned} \|f+g\| &= \sum_{l=0}^{k} \max_{t\in[a,b]} \left| (f+g)^{(l)}(t) \right| \\ &= \sum_{l=0}^{k} \max_{t\in[a,b]} \left| f^{(l)}(t) + g^{(l)}(t) \right| \\ &\leq \sum_{l=0}^{k} \max_{t\in[a,b]} \left| f^{(l)}(t) \right| + \sum_{l=0}^{k} \max_{t\in[a,b]} \left| g^{(l)}(t) \right| \\ &= ||f|| + ||g||. \end{aligned}$$

Example 1.1.20 *With* l_p*,* $p \geq 1$*, we denote the set of all sequences* $x = \{x_l\}_{l\in\mathbb{N}}$ *for which* $\sum_{l=1}^{\infty} |x_l|^p < \infty$*. Note that* l_p *is a vector space. In* l_p*,* $1 \leq p < \infty$*, we define*

$$\|x\| = \left(\sum_{l=1}^{\infty} |x_l|^p \right)^{\frac{1}{p}}, \quad x = \{x_l\}_{l\in\mathbb{N}} \in l_p. \tag{1.5}$$

We will check that (1.5) satisfies all axioms for a norm. Let $x, y \in l_p$, $x = \{x_l\}_{l\in\mathbb{N}}$, $y = \{y_l\}_{l\in\mathbb{N}}$, $\lambda \in F$ *be arbitrarily chosen. Then*

1. $\|x\| \geq 0$.
2.

$$0 \quad = \quad \|x\| \quad iff \quad \left(\sum_{l=1}^{\infty} |x_l|^p\right)^{\frac{1}{p}} = 0 \quad iff$$

$$\sum_{l=1}^{\infty} |x_l|^p = 0 \quad iff \quad x_l = 0 \quad for \quad any \quad l \in \mathbb{N} \quad iff \quad x = 0.$$

3.

$$\begin{aligned}
\|\lambda x\| &= \left(\sum_{l=1}^{\infty} |\lambda x_l|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{l=1}^{\infty} |\lambda|^p |x_l|^p\right)^{\frac{1}{p}} \\
&= |\lambda| \left(\sum_{l=1}^{\infty} |x_l|^p\right)^{\frac{1}{p}} \\
&= |\lambda| \|x\|.
\end{aligned}$$

4. *Since for any* $m \in \mathbb{N}$, *using Minkowski's inequality, we have*

$$\left(\sum_{l=1}^{m} |x_l + y_l|^p\right)^{\frac{1}{p}} \leq \left(\sum_{l=1}^{m} |x_l|^p\right)^{\frac{1}{p}} + \left(\sum_{l=1}^{m} |y_l|^p\right)^{\frac{1}{p}},$$

we conclude that

$$\left(\sum_{l=1}^{\infty} |x_l + y_l|^p\right)^{\frac{1}{p}} \leq \left(\sum_{l=1}^{\infty} |x_l|^p\right)^{\frac{1}{p}} + \left(\sum_{l=1}^{\infty} |y_l|^p\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} \|x+y\| &= \left(\sum_{l=1}^{\infty}|x_l+y_l|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{l=1}^{\infty}|x_l|^p\right)^{\frac{1}{p}}+\left(\sum_{l=1}^{\infty}|y_l|^p\right)^{\frac{1}{p}} \\ &= \|x\|+\|y\|. \end{aligned}$$

Exercise 1.1.21 *Check that*

$$\|f\| = |f(a)|+\left|f'(a)\right|+\max_{t\in[a,b]}\left|f''(t)\right|, \quad f\in\mathscr{C}^2([a,b]),$$

satisfies all axioms for a norm.

Example 1.1.22 *In* $\mathscr{C}^1([a,b])$ *we define*

$$\|f\| = \max_{t\in[a,b]}\left|f'(t)\right|. \tag{1.6}$$

Since

$$\begin{aligned} \|f\| &= 0 \quad iff \quad \max_{t\in[a,b]}\left|f'(t)\right| = 0 \quad iff \\ f'(t) &= 0 \quad for \quad any \quad t\in[a,b] \quad iff \quad f\equiv const \quad on \quad [a,b], \end{aligned}$$

(1.6) does not satisfy the axioms for a norm.

Exercise 1.1.23 *Check if*

$$\|f\| = \max_{t\in[a,b]}\left|f'(t)\right|+|f(b)-f(a)|$$

satisfies all axioms for a norm in $\mathscr{C}^1([a,b])$.

Answer. No.

Note that in a normed vector space over F, a metric can be defined by

$$d(x,y) = \|x-y\|.$$

It is evident that the defined metric satisfies all axioms for a metric. Below we will suppose that E is a normed space with a norm $\|\cdot\|$.

Lemma 1.1.24 *For every $x, y \in E$ the following inequality*

$$| \|x\| - \|y\| | \leq \|x - y\|$$

holds.

Proof 1.1.25 *We have*

$$\begin{aligned} \|x\| &= \|x - y + y\| \\ &\leq \|x - y\| + \|y\|. \end{aligned}$$

Therefore,

$$\|x\| - \|y\| \leq \|x - y\|.$$

If we interchange the positions of x and y in the last inequality, we get

$$\begin{aligned} \|y\| - \|x\| &\leq \|y - x\| \\ &= \|x - y\|. \end{aligned}$$

This completes the proof.

Definition 1.1.26

1. *An element $x_0 \in E$ will be called a limit of a sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$, if*

$$\|x_n - x_0\| \to 0, \quad as \quad n \to \infty.$$

We will write $x_n \to x_0$, as $n \to \infty$ or $\lim\limits_{n \to \infty} x_n = x_0$.

2. *For $r > 0$ the set*

$$S_r(x_0) = \{x \in E : \|x - x_0\| < r\} \, (S_r[x_0] = \{x \in E : \|x - x_0\| \leq r\})$$

will be called an open (closed) ball with a center x_0 and radius r. Sometimes, we will say that $S_r(x_0)$ is a neighborhood of x_0.

3. *A set $M \subset E$ is said to be bounded, if there exists a positive constant c such that $\|x\| \leq c$ for any $x \in M$.*

Theorem 1.1.27 *Every convergent sequence in E is a bounded sequence.*

Proof 1.1.28 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in E to the element $x_0 \in E$. Let $\varepsilon > 0$ be arbitrarily chosen and fixed. Then there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that*

$$\|x_n - x_0\| < \varepsilon$$

for any $n > N$, $n \in \mathbb{N}$. This with Lemma 1.1.24 yield

$$\|x_n\| - \|x_0\| < \varepsilon \quad or \quad \|x_n\| < \varepsilon + \|x_0\|$$

for any $n > N$, $n \in \mathbb{N}$. Let

$$c = \max\{\|x_1\|, \ldots, \|x_N\|, \varepsilon + \|x_0\|\}.$$

Then

$$\|x_n\| \leq c$$

for any $n \in \mathbb{N}$. This completes the proof.

Theorem 1.1.29 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in E to the element $x_0 \in E$.*

1. *For any $r > 0$ there is an $N = N(r) \in \mathbb{N}$ such that $x_n \in S_r(x_0)$ for any $n > N$.*
2. *Every subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x_0.*
3. *If $\{\lambda_n\}_{n\in\mathbb{N}} \subset F$ and $\lambda_n \to \lambda_0$ as $n \to \infty$, $\lambda_0 \in F$, then $\lambda_n x_n \to \lambda_0 x_0$ as $n \to \infty$.*
4. *If $\{y_n\}_{n\in\mathbb{N}} \subset E$ and $y_n \to y_0$ as $n \to \infty$, $y_0 \in E$, then $x_n + y_n \to x_0 + y_0$, as $n \to \infty$.*
5. *$\|x_n\| \to \|x_0\|$, as $n \to \infty$.*
6. *x_0 is unique.*

Proof 1.1.30

1. Let $r > 0$ *be arbitrarily chosen and fixed. Then there is an* $N = N(r) \in \mathbb{N}$ *such that*

$$\|x_n - x_0\| \leq r$$

for any $n > N$, $n \in \mathbb{N}$, *i.e.,* $x_n \in S_r(x_0)$ *for any* $n > N$, $n \in \mathbb{N}$.

2. Let $\varepsilon > 0$ *be arbitrarily chosen and fixed. Then there exists an* $N = N(\varepsilon) \in \mathbb{N}$ *such that*

$$\|x_n - x_0\| < \varepsilon \tag{1.7}$$

for any $n > N$, $n \in \mathbb{N}$. *Also, there is a* $K = K(\varepsilon) \in \mathbb{N}$ *such that* $n_k > N$ *for any* $k > K$, $k \in \mathbb{N}$. *Hence, and (1.7), we get*

$$\|x_{n_k} - x_0\| < \varepsilon$$

for any $k > K$, $k \in \mathbb{N}$.

3. Since $x_n \to x_0$ *and* $\lambda_n \to \lambda_0$ *as* $n \to \infty$, *there exist positive constants* c_1 *and* c *such that*

$$|\lambda_n| \leq c_1 \quad \textit{and} \quad \|x_n\| \leq c$$

for any $n \in \mathbb{N}$.

(a) Let $\lambda_0 = 0$. *Then*

$$\begin{aligned} \|\lambda_n x_n\| &= |\lambda_n|\|x_n\| \\ &\leq c|\lambda_n| \\ &\to 0, \quad \textit{as} \quad n \to \infty. \end{aligned}$$

(b) Let $\lambda_0 \neq 0$. *Then*

$$\begin{aligned} \|\lambda_n x_n - \lambda_0 x_0\| &= \|\lambda_n x_n - \lambda_n x_0 + \lambda_n x_0 - \lambda_0 x_0\| \\ &\leq \|\lambda_n x_n - \lambda_n x_0\| + \|\lambda_n x_0 - \lambda_0 x_0\| \\ &= |\lambda_n|\|x_n - x_0\| + |\lambda_n - \lambda_0|\|x_0\| \\ &\leq c_1\|x_n - x_0\| + |\lambda_n - \lambda_0|\|x_0\| \\ &\to 0, \quad \textit{as} \quad n \to \infty. \end{aligned}$$

4. We have

$$\begin{aligned} \|x_n + y_n - x_0 - y_0\| &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|x_n - x_0\| + \|y_n - y_0\| \\ &\to 0, \quad as \quad n \to \infty. \end{aligned}$$

5. By Lemma 1.1.24, we get

$$|\|x_n\| - \|x_0\|| \leq \|x_n - x_0\| \to 0, \quad as \quad n \to \infty.$$

6. Assume that there exists $y_0 \in E$ so that $x_n \to y_0$, as $n \to \infty$. Then

$$\begin{aligned} \|y_0 - x_0\| &= \|y_0 - x_n + x_n - x_0\| \\ &\leq \|x_n - y_0\| + \|x_n - x_0\| \\ &\to 0, \quad as \quad n \to \infty. \end{aligned}$$

Therefore, $x_0 = y_0$.

This completes the proof.

Definition 1.1.31 *A set $M \subset E$ will be called open, if for every $x_0 \in M$ there exists $r_0 > 0$ such that $S_{r_0}(x_0) \subset M$.*

Theorem 1.1.32 *Let $A_1, \ldots, A_l \subset E$ be open sets. Then $\bigcap\limits_{k=1}^{l} A_k$ is an open set in E.*

Proof 1.1.33 *Let $x \in \bigcap\limits_{k=1}^{l} A_k$ be arbitrarily chosen. Then $x \in A_k$ for any $k \in \{1, \ldots, l\}$. Since A_k, $k \in \{1, \ldots, l\}$, are open sets in E, there are $r_k > 0$ so that $S_{r_k}(x) \subset A_k$. Let $r = \min\limits_{1 \leq k \leq l} r_k$. Then $S_r(x) \subset A_k$ for any $k \in \{1, \ldots, l\}$. Therefore, $S_r(x) \subset \bigcap\limits_{k=1}^{l} A_k$. This completes the proof.*

Theorem 1.1.34 *Let* $\{A_k\}_{k\in\mathbb{N}}$ *be open sets in* E*. Then* $\bigcup\limits_{k\in\mathbb{N}} A_k$ *is an open set in* E*.*

Proof 1.1.35 *Let* $x \in \bigcup\limits_{k\in\mathbb{N}} A_k$ *be arbitrarily chosen and fixed. Then there is a* $k_0 \in \mathbb{N}$ *such that* $x \in A_{k_0}$*. Since* A_{k_0} *is an open set in* E*, there is an* $r_0 > 0$ *such that* $S_{r_0}(x) \subset A_{k_0}$*. From here,* $S_{r_0}(x) \subset \bigcup\limits_{k\in\mathbb{N}} A_k$*. This completes the proof.*

Definition 1.1.36 *A point* $a \in E$ *will be called a limit point for a set* $M \subset E$ *if for any* $r > 0$ *there is* $x \in S_r(a) \bigcap M$*,* $x \neq a$*.*

Theorem 1.1.37 *A point* $a \in E$ *is a limit point for the set* $M \subset E$ *if and only if there is a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset M$ *that converges to* a *and* $x_n \neq a$ *for any* $n \in \mathbb{N}$*.*

Proof 1.1.38

1. *Let* $a \in E$ *be a limit point for the set* M*. Then for any* $n \in \mathbb{N}$ *there are* $x_n \in S_{\frac{1}{n}}(a) \bigcap M$*,* $x_n \neq a$*. In this way, we obtain a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *such that*

$$\|x_n - a\| \to 0, \quad \text{as} \quad n \to \infty, \quad x_n \neq a,$$

i.e., $x_n \to a$ *as* $n \to \infty$ *and* $x_n \neq a$*.*

2. *Let there is a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset M$ *such that* $x_n \neq a$ *for any* $n \in \mathbb{N}$ *and* $x_n \to a$ *as* $n \to \infty$*. Hence, for any* $r > 0$ *there is an* $N = N(r) \in \mathbb{N}$ *such that* $x_n \in S_r(a)$ *for any* $n > N$ *and* $x_n \neq a$*.*

This completes the proof.

Definition 1.1.39 *A set* $M \subset E$ *is said to be closed if it contains all its limit points.*

Theorem 1.1.40 *Let* $A_1, \ldots, A_l$ *be closed sets in* E*. Then* $\bigcup\limits_{k=1}^{l} A_k$ *is a closed set in* E*.*

Proof 1.1.41 *Let* $a \in E$ *be a limit point for* $\bigcup\limits_{k=1}^{l} A_k$. *Then there exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \bigcup\limits_{k=1}^{l} A_k$ *such that* $x_n \to a$, *as* $n \to \infty$. *Hence, there is an* $m \in \{1, \ldots, l\}$ *and a subsequence* $\{x_{n_s}\}_{s\in\mathbb{N}}$ *of the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *such that* $\{x_{n_s}\}_{s\in\mathbb{N}} \subset A_m$. *We have that* $x_{n_s} \to a$, *as* $s \to \infty$ *and* $x_n \neq a$. *Hence by Theorem 1.1.37, it follows that* a *is a limit point for* A_m. *Because* A_m *is a closed set in* E, *we conclude that* $a \in A_m$. *Therefore,* $a \in \bigcup\limits_{k=1}^{l} A_k$ *and* $\bigcup\limits_{k=1}^{l} A_k$ *is a closed set in* E. *This completes the proof.*

Theorem 1.1.42 *Let* $\{A_k\}_{k\in\mathbb{N}}$ *be closed sets in* E. *Then* $\bigcap\limits_{k\in\mathbb{N}} A_k$ *is a closed set in* E.

Proof 1.1.43 *Let* $a \in E$ *be a limit point for* $\bigcap\limits_{k\in\mathbb{N}} A_k$. *Then there exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \bigcap\limits_{k\in\mathbb{N}} A_k$ *such that* $x_n \to a$, *as* $n \to \infty$. *Hence,* $\{x_n\}_{n\in\mathbb{N}} \subset A_k$, $x_n \to a$, *as* $n \to \infty$ *for any* $k \in \mathbb{N}$. *Therefore,* a *is a limit point of* A_k *for any* $k \in \mathbb{N}$. *Because* A_k, $k \in \mathbb{N}$ *are closed sets in* E, *we have that* $a \in A_k$ *for any* $k \in \mathbb{N}$. *Therefore,* $a \in \bigcap\limits_{k\in\mathbb{N}} A_k$ *and* $\bigcap\limits_{k\in\mathbb{N}} A_k$ *is a closed set in* E. *This completes the proof.*

Definition 1.1.44 *Let* $M \subset E$.

1. *The set* M *together with all of its limit points is called the closure of* M. *It will be denoted by* $\overline{M}$.

2. *The set* $E\backslash M$ *will be called the completion of the set* M *to* E.

3. *A point* $x_0 \in E$ *will be called an interior point for the set* M, *if there is an* $r > 0$ *such that* $S_r(x_0) \subset M$.

4. *A point* $x_0 \in E$ *will be called an exterior point of the set* M, *if there is an* $r > 0$ *such that* $S_r(x_0) \bigcap M = \emptyset$.

5. A point $x_0 \in E$ *will be called a boundary point of the set* M*, if for every* $r > 0$ *we have*

$$S_r(x_0)\bigcap M \neq \emptyset \quad and \quad S_r(x_0)\bigcap(E\backslash M) \neq \emptyset.$$

6. The set of all boundary points of the set M *will be called the boundary of the set* M *and it will be denoted by* ∂M.

Remark 1.1.45 *Note that we have the following possibilities.*

$$\partial M \subset M \quad or \quad \partial M\bigcap M = \emptyset \quad or \quad \partial M\bigcap M \neq \partial M.$$

Definition 1.1.46 *Two norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *in* E *will be called equivalent, if there are positive constants* c_1 *and* c_2 *such that*

$$c_1\|x\|_2 \leq \|x\|_1 \leq c_2\|x\|_2$$

for any $x \in E$*. We will write* $\|\cdot\|_1 \sim \|\cdot\|_2$.

Theorem 1.1.47 *In every finite-dimensional vector space every norms are equivalent.*

Proof 1.1.48 *Let* U *be a finite-dimensional vector space over* F*. With* $\{\phi_l\}_{l=1}^m$ *we will denote a basis in* U*. Then every* $x \in U$ *has the following representation*

$$x = \sum_{k=1}^m \xi_k\phi_k, \quad \xi_k \in F, \quad k \in \{1,\ldots,m\}.$$

In U *we define a norm*

$$||x|| = \left(\sum_{l=1}^m |\xi_l|^2\right)^{\frac{1}{2}} \quad for \quad x \in U. \tag{1.8}$$

We take an arbitrary norm $\|\cdot\|_1$ *in* U*. Let*

$$c_2 = \left(\sum_{l=1}^m \|\phi_l\|_1^2\right)^{\frac{1}{2}}.$$

Then for $x = \sum_{l=1}^{m} \xi_l \phi_l$, $\xi_l \in F$, $l \in \{1, \ldots, m\}$, *we have*

$$\begin{aligned} \|x\|_1 &= \left\| \sum_{l=1}^{m} \xi_l \phi_l \right\|_1 \\ &\leq \sum_{l=1}^{m} |\xi_l| \|\phi_l\|_1 \\ &\leq \left(\sum_{l=1}^{m} |\xi_l|^2 \right)^{\frac{1}{2}} \left(\sum_{l=1}^{m} \|\phi_l\|_1^2 \right)^{\frac{1}{2}} \\ &= c_2 \|x\|, \end{aligned}$$

i.e.,

$$\|x\|_1 \leq c_2 \|x\|. \tag{1.9}$$

On the other hand, by Lemma 1.1.24 and (1.9), we get

$$\begin{aligned} \big| \|x\|_1 - \|y\|_1 \big| &\leq \|x - y\|_1 \\ &\leq c_2 \|x - y\| \end{aligned}$$

for any $x, y \in U$. *Therefore, the function* $\|\cdot\|_1$ *is a continuous function in* U. *Hence, there exists*

$$c_1 = \inf_{\|x\|=1} \|x\|_1.$$

Consequently

$$\left\| \frac{x}{\|x\|} \right\|_1 \geq c_1 \quad or \quad \|x\|_1 \geq c_1 \|x\|.$$

This completes the proof.

Exercise 1.1.49 *Prove that (1.8) satisfies all axioms for a norm.*

Theorem 1.1.50 *Let* L *be a linear subspace of* E *which is a closed set in* E. *Then*

$$\|l\|_{E/L} = \inf_{x \in l} \|x\|, \quad l \in E/L, \tag{1.10}$$

is a norm in the quotient space E/L.

Proof 1.1.51 *Firstly, we will prove that every $l \in E/L$ is a closed set. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of l such that $x_n \to x_0$, as $n \to \infty$. We fix $m \in \mathbb{N}$ and consider $x_m - x_n$ for $n \in \mathbb{N}$. We have that $x_m - x_n \in L$ for any $n \in \mathbb{N}$ and $x_m - x_n \to x_m - x_0$, as $n \to \infty$. Hence, $x_m - x_0 \in L$. Because $x_m \in l$, we get that $x_0 \in l$.*
Let $l, m \in E/L$ be arbitrarily chosen.

1. $\|l\|_{E/L} \geq 0$.
2. *We will prove that $\|l\|_{E/L} = 0$ if and only if $l = L$.*
   - *(a) Let $\|l\|_{E/L} = 0$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of l such that $x_n \to 0$ as $n \to \infty$. Because l is a closed set, in L, we obtain that $0 \in l$ and hence $l = L$.*
   - *(b) Let $l = L$. Then $0 \in l$ and $\|l\| = 0$.*
3. *Let $\lambda \in F$ be arbitrarily chosen. Then*

$$\lambda l = \{\lambda x : x \in l\}$$

and

$$\begin{aligned}
\|\lambda l\|_{E/L} &= \inf_{x\in l} \|\lambda x\| \\
&= |\lambda| \inf_{x\in l} \|x\| \\
&= |\lambda| \|l\|_{E/L}.
\end{aligned}$$

4. *We have*

$$\begin{aligned}
\|l+m\|_{E/L} &= \inf_{x\in l+m} \|x\| \\
&\leq \inf_{\substack{x = x_1 + x_2 \\ x_1 \in l, x_2 \in m}} (\|x_1 + x_2\|) \\
&\leq \inf_{\substack{x = x_1 + x_2 \\ x_1 \in l, x_2 \in m}} (\|x_1\| + \|x_2\|) \\
&\leq \inf_{x_1 \in l} \|x_1\| + \inf_{x_2 \in m} \|x_2\| \\
&= \|l\|_{E/L} + \|m\|_{E/L}.
\end{aligned}$$

This completes the proof.

Theorem 1.1.52 *Let L be a closed linear subspace of E. Then the sequence $\{l_n\}_{n\in\mathbb{N}}$ of elements of E/L is convergent to l if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements $x_n \in l_n$ such that $x_n \to x$, as $n \to \infty$, $x \in l$.*

Proof 1.1.53

1. *Let $\{l_n\}_{n\in\mathbb{N}}$ be a sequence of elements of E/L that converges to l. Then, we get*

$$\|l_n - l\|_{E/L} \to 0, \quad as \quad n \to \infty,$$

that is

$$\|l_n - l\|_{E/L} = \varepsilon_n, \quad \varepsilon_n \to 0, \quad as \quad n \to \infty.$$

Hence, there exist $y_n \in l_n$ and $x \in l$ such that

$$\|y_n - x\| < 2\varepsilon_n.$$

Let $x_0 \in l$ be arbitrarily chosen. Then

$$\begin{aligned} \|y_n - x\| &= \|(y_n - x + x_0) - x_0\| \\ &< 2\varepsilon_n. \end{aligned}$$

Since $x_0, x \in l$, we have that $x - x_0 \in L$. Therefore,

$$x_n = y_n - x + x_0 \in l_n.$$

Consequently for every $x_0 \in l$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in l_n$, such that $x_n \to x_0$, as $n \to \infty$.

2. *Let there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in l_n$, such that $x_n \to x_0$, as $n \to \infty$, $x_0 \in l$. Then, using (1.10),*

$$\|l_n - l\| \le \|x_n - x_0\| \to 0, \quad as \quad n \to \infty.$$

This completes the proof.

Definition 1.1.54 *Let L be a linear subspace of E. We define a distance from $x \in E$ to L as follows*

$$dist(x,L) = \inf_{y \in L} \|x-y\|.$$

By Definition 1.1.54, we get

1. $dist(x,L) \geq 0$,
2. for any $y \in L$, we have

$$dist(x,L) \leq \|x-y\|,$$

3. for any $\varepsilon > 0$ there exists $y_\varepsilon \in L$ such that

$$\|x-y_\varepsilon\| < \varepsilon + dist(x,L).$$

Theorem 1.1.55 *Let L be a closed linear subspace of E. If $x \notin L$, then $dist(x,L) > 0$.*

Proof 1.1.56 *Assume that $dist(x,L) = 0$. Then there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ of elements of L such that*

$$\|y_n - x\| < \frac{1}{n}$$

for any $n \in \mathbb{N}$. Since L is closed, we get that $x \in L$, which is a contradiction. This completes the proof.

Theorem 1.1.57 *Let L be a finite-dimensional linear subspace of E. Then for any $x \in E$ there exists $x^* \in L$ such that*

$$dist\,(x,L) = \|x - x^*\|.$$

Proof 1.1.58 *Suppose that L is m-dimensional.*

1. *If $x \in L$, then $dist(x,L) = 0$ and $x = x^*$.*

2. Let $x \notin L$. Then $d = dist(x,L) > 0$. We take $\{\phi_l\}_{l=1}^m$ to be a basis in L. Then any $x \in L$ can be represented in the following way

$$y = \sum_{l=1}^{m} y_l \phi_l, \quad y_l \in F, \quad l \in \{1, \ldots, m\}.$$

Define a norm in L in the following way

$$\|y\|_c = \left(\sum_{l=1}^{m} |y_l|^2 \right)^{\frac{1}{2}} \quad for \quad y = \sum_{l=1}^{m} y_l \phi_l \in L.$$

Because L is finite-dimensional, all norms in L are equivalent. For a norm $\| \cdot \|$ in L there exist positive constants α and β such that

$$\alpha \|z\|_c \leq \|z\| \leq \beta \|z\|_c$$

for any $z \in L$. Take

$$r = \frac{d + 1 + \|x\|}{\alpha}$$

and let $y \in L$ be arbitrarily chosen. If $\|y\|_c > r$, then

$$\begin{aligned} \|x - y\| &\geq \|y\| - \|x\| \\ &\geq \alpha \|y\|_c - \|x\| \\ &> \alpha r - \|x\| \\ &= d + 1. \end{aligned}$$

Therefore, d is achieved for $\|y\| \leq r$. Since $\{y \in L : \|y\|_c \leq r\}$ is a closed and bounded set in L, and $x \mapsto \|x\|$ is a continuous function on it, there exists $x^* \in L$ such that

$$\inf_{\|y\|_c \leq r} \|x - y\| = \|x - x^*\|.$$

This completes the proof.

Definition 1.1.59 *The normed space E will be called strongly normed space if the equality*

$$\|x+y\| = \|x\| + \|y\|$$

holds if and only if $y = \lambda x$, $\lambda > 0$, $y, x \in E$.

Theorem 1.1.60 *Let E be a strongly normed space and L be a finite-dimensional linear subspace of E. If for* $x \in E$ *there exists* $x^* \in L$ *such that*

$$\|x - x^*\| = \inf_{y \in L} \|x - y\|,$$

then x^* *is unique.*

Proof 1.1.61 *If* $dist(x,L) = 0$*, then* $x = x^*$*. Suppose that* $d = dist(x,L) > 0$*. Assume that there are* $x_1^*, x_2^* \in L$ *such that*

$$\begin{aligned} d &= \|x - x_1^*\| \\ &= \|x - x_2^*\|. \end{aligned}$$

Then

$$\begin{aligned} \left\| x - \frac{x_1^* + x_2^*}{2} \right\| &= \left\| \frac{x - x_1^*}{2} + \frac{x - x_2^*}{2} \right\| \\ &\le \frac{1}{2}\|x - x_1^*\| + \frac{1}{2}\|x - x_2^*\| \\ &= \frac{1}{2}d + \frac{1}{2}d \\ &= d. \end{aligned}$$

Hence,

$$\begin{aligned} \|2x - (x_1^* + x_2^*)\| &= 2d \\ &= \|(x - x_1^*) + (x - x_2^*)\| \\ &= \|x - x_1^*\| + \|x - x_2^*\|. \end{aligned}$$

Since E is a strongly normed space, there exists $\lambda > 0$ *such that*

$$x - x_1^* = \lambda\,(x - x_2^*)\,.$$

If $\lambda \neq 1$*, then*

$$x = \frac{1}{1-\lambda}\,(x_1^* - \lambda x_2^*) \in L,$$

which is a contradiction. Therefore, $\lambda = 1$ *and* $x_1^* = x_2^*$*. This completes the proof.*

Lemma 1.1.62 (Riesz's Lemma) *Let L be a closed linear subspace of E and* $L \neq E$*. Then for any* $\varepsilon \in (0,1)$ *there exists* $z_\varepsilon \notin L$*,* $\|z_\varepsilon\| = 1$*, such that*

$$dist\,(z_\varepsilon, L) > 1 - \varepsilon.$$

Proof 1.1.63 *Since* $L \neq E$*, there exists* $x \in E$ *and* $x \notin L$*. Let* $d = \inf\limits_{y \in L} \|x - y\|$*. We have* $d > 0$*. Then for any* $\varepsilon \in (0,1)$ *there exists* $y_\varepsilon \in L$ *such that*

$$\begin{aligned} d &\leq \|y_\varepsilon - x\| \\ &< \frac{d}{1-\varepsilon}. \end{aligned}$$

Let

$$z_\varepsilon = \frac{y_\varepsilon - x}{\|y_\varepsilon - x\|}.$$

We have that $\|z_\varepsilon\| = 1$*. If we suppose that* $z_\varepsilon \in L$*, then* $y_\varepsilon - x \in L$*. Hence,* $x \in L$*, which is a contradiction. Therefore,* $z_\varepsilon \notin L$*. For* $y \in L$*, we have*

$$\begin{aligned} \|z_\varepsilon - y\| &= \left\| \frac{y_\varepsilon - x}{\|y_\varepsilon - x\|} - y \right\| \\ &= \frac{\|x - (y_\varepsilon - y\|y_\varepsilon - x\|)\|}{\|y_\varepsilon - x\|} \\ &\geq \frac{d}{\|y_\varepsilon - x\|} \\ &> 1 - \varepsilon. \end{aligned}$$

Therefore, $dist\,(z_\varepsilon, L) > 1 - \varepsilon$*. This completes the proof.*

1.2 Banach Spaces

Definition 1.2.1 *A normed vector space E that is complete in the sense of convergence in norm is called a Banach space.*

Example 1.2.2 *The space E_n is a Banach space with a norm*

$$\|x\| = \left(\sum_{l=1}^{n} \xi_l^2\right)^{\frac{1}{2}}, \quad x = (\xi_1, \ldots, \xi_n) \in E_n.$$

Example 1.2.3 *The space $\mathscr{C}([a,b])$ is a Banach space with a norm*

$$\|f\| = \max_{a \le t \le b} |f(t)|.$$

Example 1.2.4 *The vector space l_p, $p \in (1, \infty)$ is a Banach space with a norm*

$$\|x\| = \left(\sum_{l=1}^{\infty} |x_l|^p\right)^{\frac{1}{p}}.$$

Theorem 1.2.5 *Let E be a Banach space and L be a closed linear subspace in it. Then E/L is a Banach space.*

Proof 1.2.6 *Let $\{l_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in E/L. We take $x_n \in l_n$ so that*

$$\|x_n - x_m\| \le 2\|l_n - l_m\|_{E/L}.$$

In this way we get a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of E. Because E is a Banach space, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to an element $x \in E$. Let l be the class containing x. Hence, and Theorem 1.1.52, we conclude that the sequence $\{l_n\}_{n\in\mathbb{N}}$ is convergent to l. Therefore, E/L is a Banach space. This completes the proof.

Definition 1.2.7 *Let $x_1, x_2, \ldots, x_n, \ldots$ be elements of a Banach space E. An expression of the form $\sum\limits_{l=1}^{\infty} x_l$ is called a series, made up of the elements of the space E. Let $s_n = \sum\limits_{l=1}^{n} x_l$. If the sequence $\{s_n\}_{n\in\mathbb{N}}$ converges, then $\sum\limits_{l=1}^{\infty} x_l$ is said to be a convergent series.*

Theorem 1.2.8 *Let* $a_n \in F$*,* $n \in \mathbb{N}$*, and* $\sum\limits_{l=1}^{\infty} a_l$ *be a convergent series. Let also,* E *be a Banach space and* $x_n \in E$*,* $\|x_n\| \leq |a_n|$*,* $n \in \mathbb{N}$*. Then* $\sum\limits_{n=1}^{\infty} x_n$ *is a convergent series.*

Proof 1.2.9 *For any* $n, p \in \mathbb{N}$*, we have*

$$\begin{aligned} \|s_{n+p} - s_n\| &= \left\| \sum_{l=n+1}^{n+p} x_l \right\| \\ &\leq \sum_{l=n+1}^{n+p} \|x_l\| \\ &\leq \sum_{l=n+1}^{n+p} |a_l|. \end{aligned}$$

Therefore, $\{s_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* E*. Because* E *is a Banach space, we conclude that the sequence* $\{s_n\}_{n\in\mathbb{N}}$ *is convergent. This completes the proof.*

An operator is generally a mapping that acts on the elements of a vector space to produce other elements of the same or other vector space. The most common operators are linear operators, which act on vector spaces.

Suppose that X and Y are vector spaces over F.

Definition 1.2.10 *The operator* $\mathbb{A} : X \longmapsto Y$ *will be called a linear operator, if*

1. *it is additive, i.e.,*

$$\mathbb{A}(x_1 + x_2) = \mathbb{A}x_1 + \mathbb{A}x_2, \quad x_1, x_2 \in X,$$

2. *it is homogeneous, i.e.,*

$$\mathbb{A}(\lambda x) = \lambda \mathbb{A}x, \quad \lambda \in F, \quad x \in X.$$

Example 1.2.11 *Let* $K(t,s)$ *be a continuous function on the square* $0 \leq t, s \leq 1$. *For* $x \in \mathscr{C}([0,1])$, *define the operator*

$$y(t) = \int_0^1 K(t,s)x(s)ds, \quad t \in [0,1], \quad y = \mathbb{A}x.$$

Let $X = Y = \mathscr{C}([0,1])$. *It is evident that* $\mathbb{A} : X \longmapsto Y$. *We will prove that it is a linear operator.*

1. *Let* $x_1, x_2 \in X$ *be arbitrarily chosen. Then*

$$\begin{aligned} \mathbb{A}x_1(t) &= \int_0^1 K(t,s)x_1(s)ds, \\ \mathbb{A}x_2(t) &= \int_0^1 K(t,s)x_2(s)ds, \\ \mathbb{A}(x_1 + x_2)(t) &= \int_0^1 K(t,s)(x_1(s) + x_2(s))\,ds \\ &= \int_0^1 K(t,s)x_1(s)ds + \int_0^1 K(t,s)x_2(s)ds \\ &= \mathbb{A}x_1(t) + \mathbb{A}x_2(t), \quad t \in [0,1]. \end{aligned}$$

2. *Let* $\lambda \in F$ *and* $x \in X$ *be arbitrarily chosen and fixed. Then*

$$\begin{aligned} \mathbb{A}(\lambda x)(t) &= \int_0^1 K(t,s)\lambda x(s)ds \\ &= \lambda \int_0^1 K(t,s)x(s)ds \\ &= \lambda \mathbb{A}x(t), \quad t \in [0,1]. \end{aligned}$$

Therefore, $\mathbb{A} : X \longmapsto Y$ *is a linear operator.*

Example 1.2.12 *For* $x \in \mathscr{C}^1([0,1])$, *define the operator*

$$y(t) = \frac{d}{dt}x(t), \quad t \in [0,1], \quad y = \mathbb{A}x.$$

Let $X = \mathscr{C}^1([0,1])$, $Y = \mathscr{C}([0,1])$. *It is evident that* $\mathbb{A} : X \longmapsto Y$. *We will prove that it is a linear operator.*

1. Let $x_1, x_2 \in X$ be arbitrarily chosen and fixed. Then

$$\begin{aligned}
\mathbb{A}x_1(t) &= \frac{d}{dt}x_1(t),\\
\mathbb{A}x_2(t) &= \frac{d}{dt}x_2(t),\\
\mathbb{A}\left(x_1+x_2\right)(t) &= \frac{d}{dt}\left(x_1+x_2\right)(t)\\
&= \frac{d}{dt}x_1(t)+\frac{d}{dt}x_2(t)\\
&= \mathbb{A}x_1(t)+\mathbb{A}x_2(t), \quad t \in [0,1].
\end{aligned}$$

2. Let $\lambda \in F$ and $x \in X$ be arbitrarily chosen. Then

$$\begin{aligned}
\mathbb{A}\left(\lambda x\right)(t) &= \frac{d}{dt}\left(\lambda x\right)(t)\\
&= \lambda\frac{d}{dt}x(t)\\
&= \lambda\mathbb{A}x(t), \quad t \in [0,1].
\end{aligned}$$

Therefore, $\mathbb{A} : X \longmapsto Y$ is a linear operator.

Example 1.2.13 *For $x \in \mathscr{C}\left([0,1]\right)$, define the operator*

$$y = \int_0^1 (x(t))^2\,dt, \quad y = \mathbb{A}x.$$

Let $X = \mathscr{C}\left([0,1]\right)$, $Y = F$. It is evident that $\mathbb{A} : X \longmapsto Y$. Let

$$x_1(t) = 1, \quad x_2(t) = t, \quad t \in [0,1].$$

Then

$$\mathbb{A}x_1 = \int_0^1 dt$$

$$
\begin{aligned}
&= t\Big|_{t=0}^{t=1} \\
&= 1, \\
\mathbb{A}x_2 &= \int_0^1 t^2 dt \\
&= \frac{1}{3}t^3\Big|_{t=0}^{t=1} \\
&= \frac{1}{3}, \\
\mathbb{A}x_1 + \mathbb{A}x_2 &= 1 + \frac{1}{3} \\
&= \frac{4}{3}, \\
\mathbb{A}(x_1 + x_2) &= \int_0^1 (t+1)^2 dt \\
&= \int_0^1 \left(t^2 + 2t + 1\right) dt \\
&= \int_0^1 t^2 dt + 2\int_0^1 t dt + \int_0^1 dt \\
&= \frac{7}{3}.
\end{aligned}
$$

Therefore,

$$\mathbb{A}(x_1 + x_2) \neq \mathbb{A}x_1 + \mathbb{A}x_2.$$

Consequently $\mathbb{A} : X \longmapsto Y$ *is not a linear operator.*

Exercise 1.2.14 *For* $x \in \mathscr{C}([0,1])$, *define the operator*

$$y(t) = x\left(t^2\right), \quad t \in [0,1], \quad y = \mathbb{A}x.$$

Prove that $\mathbb{A} : \mathscr{C}([0,1]) \to \mathscr{C}([0,1])$ *is a linear operator.*

1.3 Linear Operators in Normed Vector Spaces

In this section, suppose that X and Y are normed vector spaces over F. The convergence in X and Y is a norm convergence.

Definition 1.3.1 *We say that the linear operator $A : X \longmapsto Y$ is continuous at $x \in X$, if for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that*

$$\|Ax_1 - Ax\| < \varepsilon$$

whenever $\|x_1 - x\| < \delta$, $x_1 \in X$. In other words, the linear operator $A : X \longmapsto Y$ is said to be continuous at $x \in X$ if $Ax_n \to Ax$ in Y, as $n \to \infty$ whenever $x_n \to x$ in X, as $n \to \infty$, where $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of elements of X. We say that the linear operator $A : X \longmapsto Y$ is continuous in X, if it is continuous at every point of X.

Example 1.3.2 *Let $X = Y = \mathscr{C}([0,1])$. Consider the operator*

$$Ax(t) = t^2 \int_0^1 x(s)ds, \quad t \in [0,1], \quad x \in X.$$

Let $x \in X$ be arbitrarily chosen and fixed. We take $\varepsilon > 0$ arbitrarily and $x_n \in X$ such that

$$\|x_n - x\| = \max_{t\in[0,1]} |x_n(t) - x(t)| < \varepsilon.$$

Hence,

$$\begin{aligned} |Ax_n(t) - Ax(t)| &= \left| t^2 \int_0^1 x_n(s)ds - t^2 \int_0^1 x(s)ds \right| \\ &= \left| t^2 \int_0^1 (x_n(s) - x(s))\, ds \right| \\ &\le t^2 \int_0^1 |x_n(s) - x(s)|\, ds \\ &\le \int_0^1 \|x_n - x\| ds \end{aligned}$$

$$< \quad \varepsilon, \quad t \in [0,1].$$

Because $\varepsilon > 0$ *was arbitrarily chosen, we conclude that* A *is continuous at* x*. Since* $x \in X$ *was arbitrarily chosen, we get that the operator* A *is continuous in* X*.*

Now we suppose that $A : X \longmapsto Y$ is a linear continuous operator. We take $x = y+z$, $y, z \in X$. Then

$$\begin{aligned} Ax &= A(y+z) \\ &= Ay + Az \\ &= Ay + A(x-y). \end{aligned}$$

Therefore,

$$A(x-y) = Ax - Ay. \tag{1.11}$$

We set $x = y$ in (1.11) and we get

$$\begin{aligned} A0 &= Ax - Ax \\ &= 0. \end{aligned}$$

We set $x = 0$ in (1.11) and we obtain

$$\begin{aligned} A(-y) &= A0 - Ay \\ &= -Ay. \end{aligned} \tag{1.12}$$

Theorem 1.3.3 *Let* $A : X \longmapsto Y$ *be linear operator, which is continuous at a single point* $x_0 \in X$*. Then it is continuous on the entire space* X*.*

Proof 1.3.4 *Let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence of elements of* X *such that* $x_n \to x$*, as* $n \to \infty$*, in* X*,* $x \in X$*. Hence,*

$$x_n - x + x_0 \to x_0, \quad as \quad n \to \infty.$$

Therefore,

$$A(x_n - x + x_0) \to Ax_0, \quad as \quad n \to \infty. \tag{1.13}$$

Since $A : X \longmapsto Y$ *is a linear operator, we get*

$$A(x_n - x + x_0) = Ax_n - Ax + Ax_0, \quad n \in \mathbb{N}.$$

From here and (1.13)*, we obtain*

$$Ax_n - Ax + Ax_0 \to Ax_0, \quad as \quad n \to \infty.$$

Therefore,

$$Ax_n \to Ax, \quad as \quad n \to \infty.$$

This completes the proof.

Definition 1.3.5 *Let* $A, B : X \longmapsto Y$ *be linear operators. We define the addition of the operators A and B by*

$$(A+B)x = Ax + Bx, \quad x \in X,$$

and the scalar multiplication by

$$(\lambda A)x = \lambda Ax, \quad x \in X, \quad \lambda \in F.$$

The zero operator O is defined by

$$Ox = 0$$

for any $x \in X$*. The identity operator I is defined by*

$$Ix = x$$

for any $x \in X$*.*

Let $A, B : X \longmapsto X$*. Define*

$$\begin{aligned} (AB)x &= A(Bx), \\ A^2x &= A(Ax), \\ A^nx &= A\left(A^{n-1}x\right), \quad n \geq 3, \quad x \in X. \end{aligned}$$

Remark 1.3.6 *If* $A, B, C : X \longmapsto Y$, *then*

$$(AB)C = A(BC),$$
$$(A+B)C = AC + BC,$$
$$C(A+B) = CA + CB.$$

In the general case, we have

$$AB \neq BA.$$

Definition 1.3.7 *Let* $A : X \longmapsto Y$ *be a linear operator. We say that the linear operator* $B : X \longmapsto Y$ *is a left inverse of the operator* A, *if*

$$BA = I.$$

We say that the linear operator $C : X \longmapsto Y$ *is a right inverse of the operator* A, *if*

$$AC = I.$$

Let $B, C : X \longmapsto Y$ *be left and right inverse, respectively, of the linear operator* $A : X \longmapsto Y$. *Then*

$$\begin{aligned} B &= BI \\ &= B(AC) \\ &= (BA)C \\ &= IC \\ &= C. \end{aligned}$$

In this case it is said that the operator A *has an inverse denoted by* A^{-1}. *Thus, if* A^{-1} *exists, we have*

$$\begin{aligned} A^{-1}A &= AA^{-1} \\ &= I. \end{aligned}$$

A linear operator $A : X \longmapsto Y$ *can have at most one inverse.*